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Some worked Landau pendulum problems

Exercise 1.1 Pendulum with support moving in circle

Attempting a mechanics problem from Landau I get a different answer. I wrote up my solution to see if I can spot either where I went wrong, or demonstrate the error, and then posted it to physicsforums. I wasn't wrong, but the text wasn't either. Here's the complete result.

§1 problem 3a of [1] is to calculate the Lagrangian of a pendulum where the point of support is moving in a circle (figure and full text for problem in this Google books reference)

Answer for Exercise 1.1

The coordinates of the mass are

$$p = ae^{i\gamma t} + ile^{i\phi}, \tag{1.1}$$

or in coordinates

$$p = (a\cos\gamma t + l\sin\phi, -a\sin\gamma t + l\cos\phi). \tag{1.2}$$

The velocity is

$$\dot{p} = (-a\gamma\sin\gamma t + l\dot{\phi}\cos\phi, -a\gamma\cos\gamma t - l\dot{\phi}\sin\phi), \tag{1.3}$$

and in the square

$$\begin{aligned}\dot{p}^2 &= a^2\gamma^2 + l^2\dot{\phi}^2 - 2a\gamma\dot{\phi}\sin\gamma t\cos\phi + 2a\gamma l\dot{\phi}\cos\gamma t\sin\phi \\ &= a^2\gamma^2 + l^2\dot{\phi}^2 + 2a\gamma l\dot{\phi}\sin(\gamma t - \phi).\end{aligned} \tag{1.4}$$

For the potential our height above the minimum is

$$\begin{aligned}h &= 2a + l - a(1 - \cos\gamma t) - l\cos\phi \\ &= a(1 + \cos\gamma t) + l(1 - \cos\phi).\end{aligned} \tag{1.5}$$

In the potential the total derivative $\cos\gamma t$ can be dropped, as can all the constant terms, leaving

$$U = -mgl\cos\phi, \tag{1.6}$$

so by the above the Lagrangian should be (after also dropping the constant term $ma^2\gamma^2/2$

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\phi}^2 + 2a\gamma l\dot{\phi}\sin(\gamma t - \phi)\right) + mgl\cos\phi. \tag{1.7}$$

This is almost the stated value in the text

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\phi}^2 + 2a\gamma^2 l\sin(\gamma t - \phi)\right) + mgl\cos\phi. \tag{1.8}$$

We have what appears to be an innocent looking typo (text putting in a γ instead of a $\dot{\phi}$), but the subsequent text also didn't make sense. That referred to the omission of the total derivative $mla\gamma\cos(\phi - \gamma t)$, which isn't even a term that I have in my result.

In the physicsforums response it was cleverly pointed out by Dickfore that 1.7 can be recast into a total derivative

$$\begin{aligned} mal\gamma\dot{\phi}\sin(\gamma t - \phi) &= mal\gamma(\dot{\phi} - \gamma)\sin(\gamma t - \phi) + mal\gamma^2\sin(\gamma t - \phi) \\ &= \frac{d}{dt}\left(mal\gamma\cos(\gamma t - \phi)\right) + mal\gamma^2\sin(\gamma t - \phi), \end{aligned} \tag{1.9}$$

which resolves the conundrum!

Exercise 1.2 Pendulum with support moving in line

This problem like the last, but with the point of suspension moving in a horizontal line $x = a\cos\gamma t$.

Answer for Exercise 1.2

Our mass point has coordinates

$$\begin{aligned} p &= a\cos\gamma t + lie^{-i\phi} \\ &= a\cos\gamma t + li(\cos\phi - i\sin\phi) \\ &= (a\cos\gamma t + l\sin\phi, l\cos\phi), \end{aligned} \tag{1.10}$$

so that the velocity is

$$\dot{p} = (-a\gamma\sin\gamma t + l\dot{\phi}\cos\phi, -l\dot{\phi}\sin\phi). \tag{1.11}$$

Our squared velocity is

$$\begin{aligned} \dot{p}^2 &= a^2\gamma^2\sin^2\gamma t + l^2\dot{\phi}^2 - 2a\gamma l\dot{\phi}\sin\gamma t\cos\phi \\ &= \frac{1}{2}a^2\gamma^2\frac{d}{dt}\left(t - \frac{1}{2\gamma}\sin 2\gamma t\right) + l^2\dot{\phi}^2 - a\gamma l\dot{\phi}(\sin(\gamma t + \phi) + \sin(\gamma t - \phi)). \end{aligned} \tag{1.12}$$

In the last term, we can reduce the sum of sines, finding a total derivative term and a remainder as in the previous problem. That is

$$\begin{aligned} \dot{\phi}(\sin(\gamma t + \phi) + \sin(\gamma t - \phi)) &= (\dot{\phi} + \gamma)\sin(\gamma t + \phi) - \gamma\sin(\gamma t + \phi) + (\dot{\phi} - \gamma)\sin(\gamma t - \phi) + \gamma\sin(\gamma t - \phi) \\ &= \frac{d}{dt}\left(-\cos(\gamma t + \phi) + \cos(\gamma t - \phi)\right) + \gamma(\sin(\gamma t - \phi) - \sin(\gamma t + \phi)) \\ &= \frac{d}{dt}\left(-\cos(\gamma t + \phi) + \cos(\gamma t - \phi)\right) - 2\gamma\cos\gamma t\sin\phi. \end{aligned} \tag{1.13}$$

Putting all the pieces together and dropping the total derivatives we have the stated solution

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\phi}^2 + 2a\gamma^2 l \cos\gamma t \sin\phi\right) + mgl\cos\phi \tag{1.14}$$

Bibliography

[1] LD Landau and EM Lifshitz. *Mechanics, vol. 1*. 1976. 1.1