Peeter Joot
peeter.joot@gmail.com

Some problems from Kittel Thermal Physics, chapter II

1.1 Motivation

Some review from the ancient second (or third?) year thermal physics course I took.

1.2 Guts

Exercise 1.1 Energy and temperature ([1] *problem 2.1*)

Suppose $g(U) = CU^{3N/2}$, where C is a constant and N is the number of particles. This form of $g(U)$ actually applies to an ideal gas.

a. Show that $U = 3Nt/2$
b. Show that $(\partial^2\sigma/\partial U^2)_N$ is negative.

Answer for Exercise 1.1

Part a. Temperature We've got

$$\begin{aligned}\frac{1}{\tau} &= \frac{\partial \sigma}{\partial U} \\ &= \frac{\partial}{\partial U}\left(\ln C + \frac{3N}{2}\ln U\right) \\ &= \frac{3N}{2}\frac{1}{U},\end{aligned} \tag{1.1}$$

or

$$U = \frac{3N}{2}\tau. \tag{1.2}$$

Part b. Second derivative of entropy From above

$$\frac{\partial^2 \sigma}{\partial U^2} = -\frac{3N}{2}\frac{1}{U^2}. \tag{1.3}$$

This doesn't seem particularly suprising if we look at the plots. For example for $C = 1$ and $3N/2 = 1$ we have fig. 1.1.

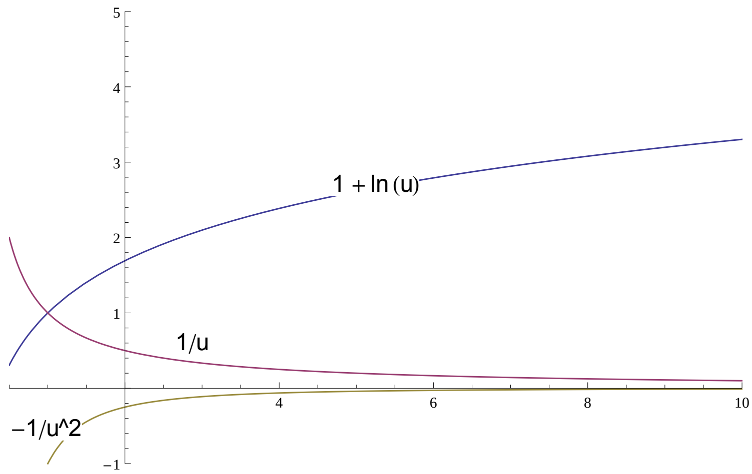

Figure 1.1: Plots of entropy and its derivatives for this multiplicity function

The rate of change of entropy with energy decreases monotonically and is always positive, but always has a negative slope.

Exercise 1.2 Paramagnetism ([1] *problem 2.2*)

Find the equilibrium value at temperrature τ of the fractional magnetization

$$\frac{M}{Nm} = \frac{2\langle s \rangle}{N} \tag{1.4}$$

of the system of N spins each of magnetic moment m in a magnetic field B. The spin excess is $2s$. Take the entropy as the logarithm of the multiplicity $g(N,s)$ as given in (1.35):

$$\sigma(s) \approx \ln g(N,0) - \frac{2s^2}{N}, \tag{1.5}$$

for $|s| \ll N$. Hint: Show that in this approximation

$$\sigma(U) = \sigma_0 - \frac{U^2}{2m^2B^2N}, \tag{1.6}$$

with $\sigma_0 = \ln g(N,0)$. Further, show that $1/\tau = -U/(m^2B^2N)$, where U denotes $\langle U \rangle$, the thermal average energy.

Answer for Exercise 1.2

I found this problem very hard to interpret. What exactly is being asked for? Equation (1.35) in the text was

$$g(N,s) \approx g(N,0)e^{-\frac{2s^2}{N}} \tag{1.7a}$$

$$g(N,0) \approx \sqrt{\frac{2}{\pi N}} 2^N, \tag{1.7b}$$

from which we find the entropy 1.5 directly after taking logarithms. The temperature is found directly

The magnetization, for a system that has spin excess $2s$ was defined as

$$U = -2smB \equiv -MB \tag{1.8}$$

and we can substitute that for s

$$\sigma(U) = \sigma_0 - \frac{U^2}{2m^2 B^2 N}, \tag{1.9}$$

and take derivatives for the temperature

$$\begin{aligned} \frac{1}{\tau} &= \frac{\partial \sigma}{\partial U} \\ &= \frac{\partial}{\partial U}\left(\sigma_0 - \frac{U^2}{2m^2 B^2 N}\right) \\ &= -\frac{U}{m^2 B^2 N} \end{aligned} \tag{1.10}$$

This gives us a relation between temperature and the energy of the system with spin excess $2s$, and we could write

$$\frac{M}{Nm} = -\frac{U}{BNm} = \frac{mB}{\tau}. \tag{1.11}$$

Is this the relation that this problem was asking for?

Two things I don't understand from this problem:

1. Where does $2\langle s \rangle / N$ come from? If we calculate the expectatation of the spin excess, we find that it is zero

$$\begin{aligned} \langle 2s \rangle &= \frac{\sqrt{\frac{2}{\pi N}} 2^N \int_{-\infty}^{\infty} ds 2s e^{-\frac{2s^2}{N}}}{2^N} \\ &= 0. \end{aligned} \tag{1.12}$$

2. If $2\langle s \rangle$ has a non-zero value, then doesn't that make $\langle U \rangle$ also zero? It seems to me that U in 1.10 is the energy of a system with spin excess s, and not any sort of average energy?

Exercise 1.3 Quantum harmonic oscillator ([1] *problem 2.3*)

a. Entropy. Find the entropy of a set of N oscillators of frequency ω as a function of the total quantum number n. Use the multiplicity function (1.55) and make the Stirling approximation $\ln N! \approx N \ln N - N$. Replace $N - 1$ by N.

b. Planck Energy. Let U denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy as $\sigma(U, N)$. Show that the total energy at temperature τ is

$$U = \frac{N\hbar\omega}{\exp\left(\hbar\omega/\tau\right) - 1} \tag{1.13}$$

This is the Planck result; it is derived again in Chapter 4 by a powerful method that does not require us to find the multiplicity function.

Answer for Exercise 1.3

Part a. Entropy The multiplicity was found in the text to be

$$g(N, n) = \frac{(N + n - 1)!}{n!\,(N - 1)!} \tag{1.14}$$

I wasn't actually able to follow the argument in the text, and found the purely combinatoric wikipedia argument [3] much clearer. A similar diagram and argument can also be found in [2] §3.8.

Taking logarithms and applying the Stirling approximation, our entropy is

$$\begin{aligned}
\sigma &= \ln g \\
&= \ln(N + n - 1)! - \ln(N - 1)! - \ln n! \\
&\approx (N + n - 1)\ln(N + n - 1) - (N + n - 1) - (N - 1)\ln(N - 1) + (N - 1) - n\ln n + n \\
&= (N - 1)\ln\frac{N + n - 1}{N - 1} + n\ln\frac{N + n - 1}{n}
\end{aligned} \tag{1.15}$$

Part b. Planck Energy Now we make the $N - 1 \rightarrow N$ replacement suggested in the problem (ie. assuming $N \gg 1$), for

$$\begin{aligned}
\sigma &\approx N\ln\frac{N + n}{N} + n\ln\frac{N + n}{n} \\
&= (N + n)\ln(N + n) - N\ln N - n\ln n \\
&= \left(N + \frac{U}{\hbar\omega}\right)\ln\left(N + \frac{U}{\hbar\omega}\right) - N\ln N - \frac{U}{\hbar\omega}\ln\frac{U}{\hbar\omega}
\end{aligned} \tag{1.16}$$

With $(x\ln x)' = \ln x + 1$, we have

$$\begin{aligned}
\frac{1}{\tau} &= \frac{\partial\sigma}{\partial U} \\
&= \frac{1}{\hbar\omega}\left(\ln\left(N + \frac{U}{\hbar\omega}\right) - 1 - \ln\frac{U}{\hbar\omega} + 1\right),
\end{aligned} \tag{1.17}$$

or

$$Ue^{\frac{\hbar\omega}{\tau}} = N\hbar\omega + U. \tag{1.18}$$

A final rearrangement gives us the Planck result 1.13.

Bibliography

[1] C. Kittel and H. Kroemer. *Thermal physics*. WH Freeman, 1980.

[2] RK Pathria. *Statistical mechanics*. Butterworth Heinemann, Oxford, UK, 1996.

[3] Wikipedia. *Einstein solid — Wikipedia, The Free Encyclopedia*, 2012. URL http://en.wikipedia.org/w/index.php?title=Einstein_solid&oldid=530449869. [Online; accessed 2-January-2013].