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1D pendulum problem in phase space

Problem 2.6 in [1] asks for some analysis of the (presumably small angle) pendulum problem in phase space, including an integration of the phase space volume energy and period of the system to the area A included within a phase space trajectory. With coordinates as in fig. 1.1, our Lagrangian is

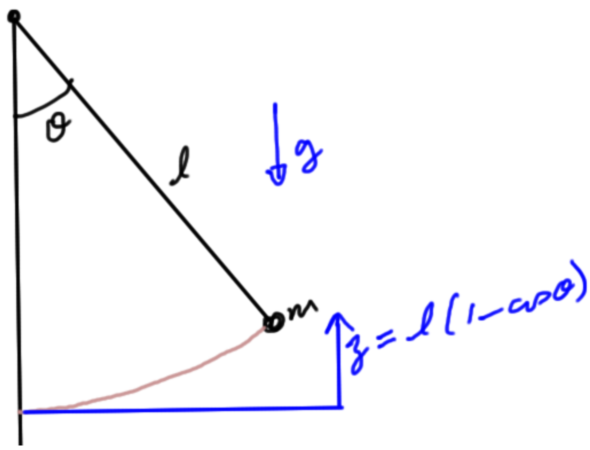

Figure 1.1: 1d pendulum

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 - g m l (1 - \cos\theta). \tag{1.1}$$

As a sign check we find for small θ from the Euler-Lagrange equations $\ddot{\theta} = -(g/l)\theta$ as expected. For the Hamiltonian, we need the canonical momentum

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta}. \tag{1.2}$$

Observe that this canonical momentum does not have dimensions of momentum, but that of angular momentum ($m l \dot{\theta} \times l$).

Our Hamiltonian is

$$H = \frac{1}{2 m l^2} p_\theta^2 + g m l (1 - \cos\theta). \tag{1.3}$$

Hamilton's equations for this system, in matrix form are

$$\frac{d}{dt}\begin{bmatrix}\theta \\ p_\theta\end{bmatrix} = \begin{bmatrix}\frac{\partial H}{\partial p_\theta} \\ -\frac{\partial H}{\partial \theta}\end{bmatrix} = \begin{bmatrix}p_\theta/ml^2 \\ -gml\sin\theta\end{bmatrix} \tag{1.4}$$

With $\omega = g/l$, it is convient to non-dimensionalize this

$$\frac{d}{dt}\begin{bmatrix}\theta \\ p_\theta/\omega ml^2\end{bmatrix} = \omega\begin{bmatrix}p_\theta/\omega ml^2 \\ -\sin\theta\end{bmatrix}. \tag{1.5}$$

Now we can make the small angle approximation. Writing

$$\mathbf{u} = \begin{bmatrix}\theta \\ p_\theta/\omega ml^2\end{bmatrix} \tag{1.6a}$$

$$i = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \tag{1.6b}$$

Our pendulum equation is reduced to

$$\mathbf{u}' = i\omega\mathbf{u}, \tag{1.7}$$

With a solution that we can read off by inspection

$$\mathbf{u} = e^{i\omega t}\mathbf{u}_0 = \begin{bmatrix}\cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t\end{bmatrix}\mathbf{u}_0 \tag{1.8}$$

Let's put the initial phase space point into polar form

$$\begin{aligned}\mathbf{u}_0^2 &= \theta_0^2 + \frac{p_0^2}{\omega^2 m^2 l^4} \\ &= \frac{2}{\omega^2 ml^2}\left(\frac{p_0^2}{2ml^2} + \frac{1}{2}\omega^2 ml^2\theta_0^2\right) \\ &= \frac{2}{gml}\left(\frac{p_0^2}{2ml^2} + \frac{1}{2}gml\theta_0^2\right)\end{aligned} \tag{1.9}$$

This doesn't appear to be an exact match for eq. (1.3), but we can write for small θ_0

$$\begin{aligned}1 - \cos\theta_0 &= 2\sin^2\left(\frac{\theta_0}{2}\right) \\ &\approx 2\left(\frac{\theta_0}{2}\right)^2 \\ &= \frac{\theta_0^2}{2}.\end{aligned} \tag{1.10}$$

This shows that we can rewrite our initial conditions as

$$
\mathbf{u}_0 = \sqrt{\frac{2E}{gml}} e^{i\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1.11}
$$

where

$$
\tan\phi = \left(\omega m l^2 \theta_0 / p_0\right). \tag{1.12}
$$

Our time evolution in phase space is given by

$$
\begin{bmatrix} \theta(t) \\ p_\theta(t) \end{bmatrix} = \sqrt{\frac{2E}{gml}} \begin{bmatrix} \cos(\omega t + \phi) \\ -\omega m l^2 \sin(\omega t + \phi) \end{bmatrix}, \tag{1.13}
$$

or

$$
\boxed{\begin{bmatrix} \theta(t) \\ p_\theta(t) \end{bmatrix} = \frac{1}{\omega l} \sqrt{\frac{2E}{m}} \begin{bmatrix} \cos(\omega t + \phi) \\ -\omega m l^2 \sin(\omega t + \phi) \end{bmatrix}.} \tag{1.14}
$$

This is plotted in fig. 1.2.

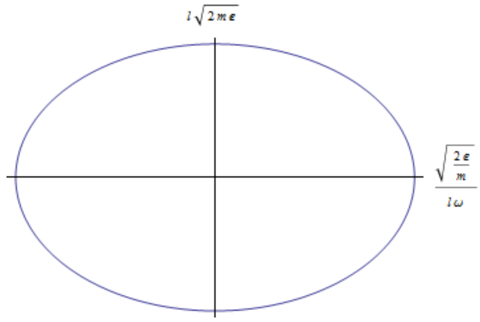

Figure 1.2: Phase space trajectory for small angle pendulum

The area of this ellipse is

$$
\begin{aligned}
A &= \pi \frac{1}{\omega^2 l^2} \frac{2E}{m} \omega m l^2 \\
&= \frac{2\pi}{\omega} E.
\end{aligned} \tag{1.15}
$$

With τ for the period of the trajectory, this is

$$
A = \tau E. \tag{1.16}
$$

As a final note, observe that the oriented integral from problem 2.5 of the text $\oint p_\theta d\theta$, is also this area. This is a general property, which can be seen geometrically in fig. 1.3, where we see that the counterclockwise oriented integral of $\oint pdq$ would give the negative area. The integrals along the c_4, c_1 paths give the area under the blob, whereas the integrals along the other paths where the sense

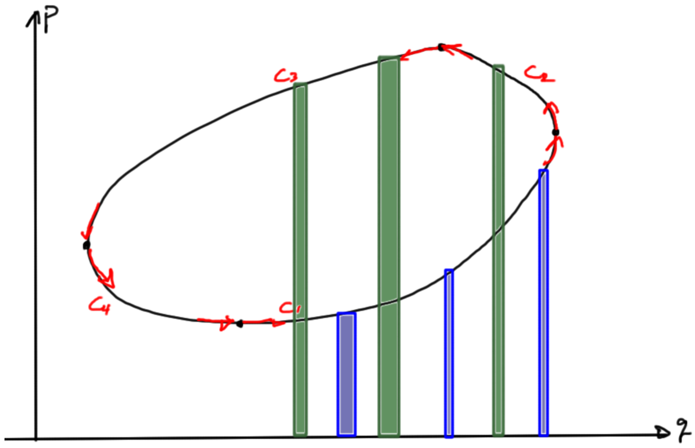

Figure 1.3: Area from oriented integral along path

is opposite, give the complete area under the top boundary. Since they are oppositely sensed, adding them gives just the area of the blob.

Let's do this $\oint p_\theta d\theta$ integral for the pendulum phase trajectories. With

$$\theta = \frac{1}{\omega l}\sqrt{\frac{2E}{m}}\cos(\omega t + \phi) \tag{1.17a}$$

$$p_\theta = -ml\sqrt{\frac{2E}{m}}\sin(\omega t + \phi) \tag{1.17b}$$

We have

$$\begin{aligned}
\oint p_\theta d\theta &= \frac{ml}{\omega l}\frac{2E}{m}\int_0^{2\pi/\omega} \sin^2(\omega t + \phi)\omega dt \\
&= 2E\int_0^{2\pi/\omega} \frac{1 - \cos\left(2(\omega t + \phi)\right)}{2} dt \\
&= E\frac{2\pi}{\omega} \\
&= E\tau.
\end{aligned} \tag{1.18}$$

Bibliography

[1] RK Pathria. *Statistical mechanics*. Butterworth Heinemann, Oxford, UK, 1996. 1