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Cartesian to spherical change of variables in 3d phase space

Exercise 1.1 Cartesian to spherical change of variables in 3d phase space

[1] problem 2.2 (a). Try a spherical change of vars to verify explicitly that phase space volume is preserved.

Answer for Exercise 1.1

Our kinetic Lagrangian in spherical coordinates is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} m \left(\dot{r} \hat{\mathbf{r}} + r \sin\theta \dot{\phi} \hat{\boldsymbol{\phi}} + r \dot{\theta} \hat{\boldsymbol{\theta}} \right)^2 \\ &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\theta \dot{\phi}^2 + r^2 \dot{\theta}^2 \right)^2\end{aligned} \tag{1.1}$$

We read off our canonical momentum

$$p_r = \frac{\partial \mathcal{L}}{\partial r} = m \dot{r} \tag{1.2a}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = m r^2 \dot{\theta} \tag{1.2b}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = m r^2 \sin^2\theta \dot{\phi}, \tag{1.2c}$$

and can now express the Hamiltonian in spherical coordinates

$$\begin{aligned}H &= \frac{1}{2} m \left(\left(\frac{p_r}{m} \right)^2 + r^2 \sin^2\theta \left(\frac{p_\phi}{m r^2 \sin^2\theta} \right) + r^2 \left(\frac{p_\theta}{m r^2} \right) \right) \\ &= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2 m r^2 \sin^2\theta} + \frac{p_\theta^2}{2 m r^2}\end{aligned} \tag{1.3}$$

Now we want to do a change of variables. The coordinates transform as

$$x = r \sin\theta \cos\phi \tag{1.4a}$$

$$y = r \sin\theta \sin\phi \tag{1.4b}$$

$$z = r\cos\theta, \tag{1.4c}$$

or

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.5a}$$

$$\theta = \arccos(z/r) \tag{1.5b}$$

$$\phi = \arctan(y/x). \tag{1.5c}$$

It's not too hard to calculate the change of variables for the momenta (verified in sphericalPhaseSpaceChangeOfVars.nb). We have

$$p_r = \frac{x p_x + y p_y + z p_z}{\sqrt{x^2 + y^2 + z^2}} \tag{1.6a}$$

$$p_\theta = \frac{(p_x x + p_y y)z - p_z(x^2 + y^2)}{\sqrt{x^2 + y^2}} \tag{1.6b}$$

$$p_\phi = x p_y - y p_x \tag{1.6c}$$

Now let's compute the volume element in spherical coordinates. This is

$$
\begin{aligned}
d\omega &= dr d\theta d\phi p_r p_\theta p_\phi \qquad (1.7)\\
&= \frac{\partial(r,\theta,\phi,p_r,p_\theta,p_\phi)}{\partial(x,y,z,p_x,p_y,p_z)} dx dy dz dp_x dp_y dp_z \\
&= \begin{vmatrix}
\frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} & 0 & 0 & 0 \\
\frac{xz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)} & \frac{yz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)} & -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} & 0 & 0 & 0 \\
-\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 & 0 & 0 & 0 \\
\frac{\left(y^2+z^2\right)p_x - xyp_y - xzp_z}{(x^2+y^2+z^2)^{3/2}} & \frac{\left(x^2+z^2\right)p_y - y(xp_x+zp_z)}{(x^2+y^2+z^2)^{3/2}} & \frac{\left(x^2+y^2\right)p_z - z\left(xp_x+yp_y\right)}{(x^2+y^2+z^2)^{3/2}} & \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\
\frac{yz\left(yp_x - xp_y\right) - x\left(x^2+y^2\right)p_z}{(x^2+y^2)^{3/2}} & \frac{xz\left(xp_y - yp_x\right) - y\left(x^2+y^2\right)p_z}{(x^2+y^2)^{3/2}} & \frac{xp_x+yp_y}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \\
p_y & -p_x & 0 & -y & x & 0
\end{vmatrix} dx dy dz dp_x dp_y dp_z \\
&= dx dy dz dp_x dp_y dp_z
\end{aligned}
$$

This also has a unit determinant, as we found in the similar cylindrical change of phase space variables.

Bibliography

[1] RK Pathria. *Statistical mechanics*. Butterworth Heinemann, Oxford, UK, 1996. 1.1