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Relativisitic density of states

Setup For photons and high velocity particles our non-relativisitic density of states is insufficient. Let's redo these calculations for particles for which the energy is given by

$$\epsilon = \sqrt{\left(mc^2\right)^2 + (pc)^2}. \tag{1.1}$$

We want to convert a sum over momentum values to an energy integral

$$\begin{aligned}\mathcal{D}_3(\epsilon) &= \sum_{\mathbf{p}} \delta(\epsilon - \epsilon_{\mathbf{p}}) \\ &\rightarrow L^d \int \frac{d^d \mathbf{k}}{(2\pi)^d} \delta(\epsilon - \epsilon_{\mathbf{p}}) \\ &= L^d \int \frac{d^3 \mathbf{p}}{(2\pi\hbar)^d} \delta(\epsilon - \epsilon_{\mathbf{p}}) \\ &= L^d \int \frac{d^d (c\mathbf{p})}{(ch)^d} \delta(\epsilon - \epsilon_{\mathbf{p}}).\end{aligned} \tag{1.2}$$

Now we want to use

$$\delta(g(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{\left|g'(x)\right|_{x=x_0}}, \tag{1.3}$$

where x_0 are the roots of $g(x)$. With

$$g(cp) = \epsilon - \sqrt{\left(mc^2\right)^2 + (cp)^2}. \tag{1.4}$$

Writing p^* for the roots we have

$$cp^* = \sqrt{\epsilon^2 - \left(mc^2\right)^2}. \tag{1.5}$$

Note that

$$\sqrt{\left(mc^2\right)^2 + (cp^*)^2} = \sqrt{\epsilon^2} = \epsilon. \tag{1.6}$$

we have

$$\begin{aligned} \left|g'(cp)\right|_{p=p^*} &= \frac{1}{2}\frac{2(cp^*)}{\sqrt{\left(mc^2\right)^2 + (cp^*)^2}} \\ &= \frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}. \end{aligned} \tag{1.7}$$

3D case We can now evaluate the density of states, and do the 3D case first. We have

$$\mathcal{D}_3(\epsilon) = \frac{V}{(ch)^3}\int_0^\infty 4\pi (cp)^2 d(cp) \left(\delta\left(cp - \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right) + \delta\left(cp + \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right)\right)\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}. \tag{1.8}$$

Observe that in the switch to spherical coordinates in momentum space, our integration is now over a "radius" of momentum space, requiring just integration over the positive values. This will kill off one of our delta functions, leaving just

$$\mathcal{D}_3(\epsilon) = \frac{4\pi V}{(ch)^3}\left(\epsilon^2 - \left(mc^2\right)^2\right)\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}, \tag{1.9}$$

or

$$\boxed{\mathcal{D}_3(\epsilon) = \frac{4\pi V}{(ch)^3}\frac{\left(\epsilon^2 - \left(mc^2\right)^2\right)^{3/2}}{\epsilon}.} \tag{1.10}$$

In particular, for very high energy particles where $\epsilon \gg \left(mc^2\right)$, our 3D density of states is

$$\boxed{\mathcal{D}_3(\epsilon) \approx \frac{4\pi V}{(ch)^3}\epsilon^2} \tag{1.11}$$

This is also the desired result for photons or other massless particles.

2D case For 2D we have

$$\mathcal{D}_2(\epsilon) = \frac{A}{(ch)^2}\int_0^\infty 2\pi |cp| d(cp) \left(\delta\left(cp - \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right) + \delta\left(cp + \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right)\right)\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}. \tag{1.12}$$

Note again that we are dealing with a "radius" over this shell of momentum space volume. This is a strictly positive value. That and the corresponding integration range is important in this case since including the negative range of cp would kill the entire density function because of the pair of delta functions. That wasn't the case in 3D, where it would have resulted in an off by two error instead. Continuing the evaluation we have

$$\mathcal{D}_2(\epsilon) = \frac{2\pi A}{(ch)^2}\sqrt{\epsilon^2 - \left(mc^2\right)^2}\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}, \tag{1.13}$$

or

$$\boxed{\mathcal{D}_2(\epsilon) = \frac{2\pi A}{(ch)^2}\frac{\epsilon^2 - \left(mc^2\right)^2}{\epsilon}.} \tag{1.14}$$

For an extreme relativisitic gas where $\epsilon \gg mc^2$ (or photons where $m = 0$), we have

$$\boxed{\mathcal{D}_2(\epsilon) \approx \frac{2\pi A}{(ch)^2}\epsilon.} \tag{1.15}$$

1D case

$$\mathcal{D}_1(\epsilon) = \frac{L}{ch}\int d(cp)\left(\delta\left(cp - \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right) + \delta\left(cp + \sqrt{\epsilon^2 - \left(mc^2\right)^2}\right)\right)\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon}. \tag{1.16}$$

Question : For the 1D case, we don't have to make a switch to spherical or cylindrical coordinates, so it looks like the second delta function has to be included, and the integration range over both positive and negative values of cp?

Assuming that's the case, we have

$$\boxed{\mathcal{D}_1(\epsilon) = \frac{2L}{ch}\frac{\sqrt{\epsilon^2 - \left(mc^2\right)^2}}{\epsilon},} \tag{1.17}$$

and for $\epsilon \gg mc^2$ or $m = 0$

$$\boxed{\mathcal{D}_1(\epsilon) = \frac{2L}{ch}.} \tag{1.18}$$