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Ultra relativisitic spin zero condensation temperature

Here's a bash at one of the exam questions, where I get the time to think things through properly. I think I did something like this on the exam itself, but may have also made some arithmetic errors.

Exercise 1.1 Ultra relativisitic spin zero condensation temperature (*2013 final exam pr 2*)

Consider a Bose gas with particles having no spin and obeying an ultra relativisitic dispersion $E_{\mathbf{k}} = c|\mathbf{k}|$. Unlike photons or phonons, these particles are **conserved**, and hence we must determine the chemical potential μ which fixes their density. Working in three dimensions, show whether or not these particles will exhibit Bose condensation, and find T_c if it is nonzero.

Answer for Exercise 1.1

For the number of particles in the gas, as with photons, we still have

$$
\begin{aligned}
\langle N \rangle &= \sum_{\mathbf{k}} \frac{1}{z^{-1}e^{\beta\epsilon_{\mathbf{k}}} - 1} \\
&= \frac{1}{z^{-1} - 1} + \sum_{\mathbf{k} \neq 0} \frac{1}{z^{-1}e^{\beta\epsilon_{\mathbf{k}}} - 1}.
\end{aligned}
\tag{1.1}
$$

As in the discussion of low velocity particles in [1] §7.1, the ground state term has been split out, before making any continuum approximation of the sum over the energetic states.

Writing

$$
\langle N \rangle = N_0 + N_e,
\tag{1.2}
$$

where the number of particles in the ground state is chemical potential and temperature dependent

$$
N_0 = \frac{z}{1 - z}.
\tag{1.3}
$$

We proceed with the continuum approximation for the number of particles in the energetic states

$$N_e = \sum_{\mathbf{k} \neq 0} \frac{1}{z^{-1}e^{\beta \epsilon_{\mathbf{k}}} - 1}$$

$$\sim V \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{z^{-1}e^{\beta \epsilon_{\mathbf{k}}} - 1}$$

$$= \frac{4\pi V}{(2\pi)^3} \int_0^\infty k^2 dk \frac{1}{z^{-1}e^{\beta c k} - 1} \tag{1.4}$$

$$= \frac{V}{2\pi^2} \left(\frac{1}{\beta c}\right)^3 \int_0^\infty x^2 dx \frac{1}{z^{-1}e^x - 1}$$

$$= \frac{V}{2\pi^2} \left(\frac{1}{\beta c}\right)^3 \Gamma(3) g_3(z).$$

So we have

$$N_e = \frac{V}{\pi^2} \left(\frac{k_B T}{c}\right)^3 g_3(z) \leq \frac{V}{\pi^2} \left(\frac{k_B T}{c}\right)^3 \zeta(3). \tag{1.5}$$

Note that $\zeta(3) \approx 1.20206$, a fixed number. The key feature of Bose condensation remains. There is a finite limit to the number of particles that can be in the energetic state at a given temperature and volume. Any remaining particles are forced into the ground state.

In general the number of particles in the ground state is

$$N_0 = N - \frac{V}{\pi^2} \left(\frac{k_B T}{c}\right)^3 g_3(z), \tag{1.6}$$

and we will necessarily have particles in this state if

$$N - \frac{V}{\pi^2} \left(\frac{k_B T}{c}\right)^3 \zeta(3) > 0. \tag{1.7}$$

That temperature threshold $T \leq T_c$ is the Bose condensation temperature

$$\boxed{k_B T_c = c \left(\frac{n\pi^2}{\zeta(3)}\right)^{1/3}.} \tag{1.8}$$

With $n = N/V$, $n_0 = N_0/V$, we have for the ground state average number density

$$n_0 = n \left(1 - \frac{g_3(z)}{\zeta(3)} \left(\frac{T}{T_c}\right)^3\right) \tag{1.9}$$
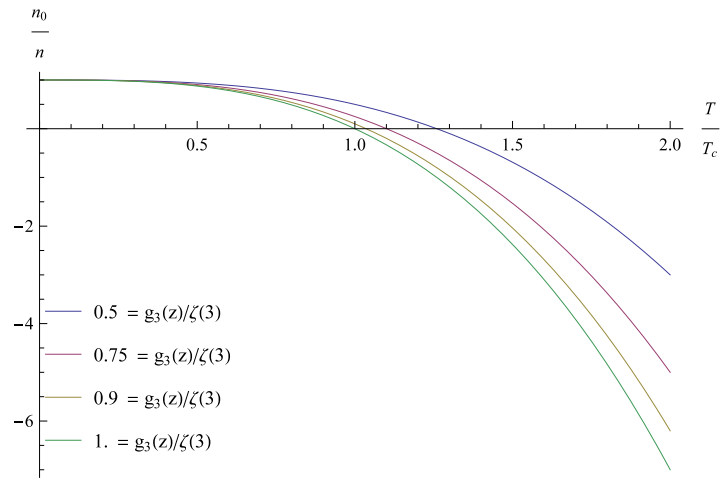
This is plotted in fig. 1.1.

Figure 1.1: Ratio of ground state number density to total number density

From the figure it appears that the notion of any sort of absolute condensation temperature is an approximation. We can start having particles go into the ground state at higher temperatures than T_c, but once the chemical potential starts approaching zero, that temperature for which we start having particles in the ground state approaches T_c. The key takeout idea appears to be, once the temperature does drop below T_c, we necessarily start having a non-zero ground state population, and as the temperature drops more and more, the ratio of the number of particles in the ground state relative to the total approaches unity (all particles are forced into the ground state).

Bibliography

[1] RK Pathria. *Statistical mechanics*. Butterworth Heinemann, Oxford, UK, 1996. 1