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Discrete Fourier Transform

In [2] a verification of the discrete Fourier transform pairs was performed. A much different looking discrete Fourier transform pair is given in [1] §A.4. This transform pair samples the points at what are called the Nykvist time instants given by

$$t_k = \frac{Tk}{2N+1}, \qquad k \in [-N, \cdots N] \tag{1.1}$$

Note that the endpoints of these sampling points are not $\pm T/2$, but are instead at

$$\pm \frac{T}{2} \frac{1}{1 + 1/N}, \tag{1.2}$$

which are slightly within the interior of the $[-T/2, T/2]$ range of interest. The reason for this slightly odd seeming selection of sampling times becomes clear if one calculate the inversion relations.

Given a periodic ($\omega_0 T = 2\pi$) bandwith limited signal evaluated only at the Nykvist times t_k,

$$\boxed{x(t_k) = \sum_{n=-N}^{N} X_n e^{j n \omega_0 t_k},} \tag{1.3}$$

assume that an inversion relation can be found. To find X_n evaluate the sum

$$\begin{aligned} \sum_{k=-N}^{N} x(t_k) e^{-j m \omega_0 t_k} &= \sum_{k=-N}^{N} \left(\sum_{n=-N}^{N} X_n e^{j n \omega_0 t_k} \right) e^{-j m \omega_0 t_k} \\ &= \sum_{n=-N}^{N} X_n \sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k} \end{aligned} \tag{1.4}$$

This interior sum has the value $2N+1$ when $n = m$. For $n \ne m$, and $a = e^{j(n-m)\frac{2\pi}{2N+1}}$, this is

$$
\begin{aligned}
\sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k} &= \sum_{k=-N}^{N} e^{j(n-m)\omega_0 \frac{Tk}{2N+1}} \\
&= \sum_{k=-N}^{N} a^k \\
&= a^{-N} \sum_{k=-N}^{N} a^{k+N} \\
&= a^{-N} \sum_{r=0}^{2N} a^r \\
&= a^{-N} \frac{a^{2N+1}-1}{a-1}.
\end{aligned} \tag{1.5}
$$

Since $a^{2N+1} = e^{2\pi j(n-m)} = 1$, this sum is zero when $n \neq m$. This means that

$$
\sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k} = (2N+1)\delta_{n,m}, \tag{1.6}
$$

which provides the desired Fourier inversion relation

$$
\boxed{X_m = \frac{1}{2N+1} \sum_{k=-N}^{N} x(t_k) e^{-jm\omega_0 t_k}.} \tag{1.7}
$$

Bibliography

[1] Franco Giannini and Giorgio Leuzzi. *Nonlinear Microwave Circuit Design*. Wiley Online Library, 2004. 1

[2] Peeter Joot. *Condensed matter physics.*, chapter Discrete Fourier transform. 2013. URL http://peeterjoot.com/archives/math2013/phy487.pdf. [Online; accessed 02-December-2014]. 1