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ECE1254H Modeling of Multiphysics Systems. Lecture 4: Modified nodal analysis. Taught by Prof. Piero Triverio

1.0.1 *Disclaimer*

Peeter's lecture notes from class. These may be incoherent and rough.

1.0.2 *Modified nodal analysis*

We add extra unknowns for

- branch currents for voltage sources
- all elements for which it is impossible or inconvieient to write $i = f(v_1, v_2)$.
  Imagine, for example, that we have a component illustrated in fig. 1.1.

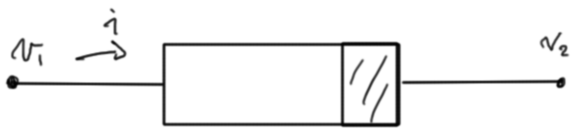

Figure 1.1: Variable voltage device

$$v_1 - v_2 = 3i^2 \tag{1.1}$$

- any current which is controlling dependent sources, as in fig. 1.2.
- Inductors

$$v_1 - v_2 = L\frac{di}{dt}. \tag{1.2}$$

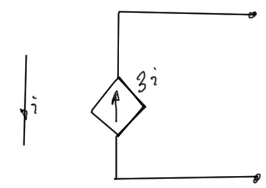

Figure 1.2: Current controlled device

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Solving large systems

We are interested in solving linear systems of the form

$$M\bar{x} = \bar{b}, \tag{2.1}$$

possibly with thousands of elements.

2.1 Gaussian elimination

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2.2}$$

It's claimed for now, to be seen later, that back substitution is the fastest way to arrive at the solution, less computationally complex than completion the diagonalization.

Steps

$$(1) \cdot \frac{M_{21}}{M_{11}} \implies \begin{bmatrix} M_{21} & \frac{M_{21}}{M_{11}} M_{12} & \frac{M_{21}}{M_{11}} M_{13} \end{bmatrix} \tag{2.3}$$

$$(2) \cdot \frac{M_{31}}{M_{11}} \implies \begin{bmatrix} M_{31} & \frac{M_{31}}{M_{11}} M_{32} & \frac{M_{31}}{M_{11}} M_{33} \end{bmatrix} \tag{2.4}$$

This gives

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} - \frac{M_{21}}{M_{11}} M_{12} & M_{23} - \frac{M_{21}}{M_{11}} M_{13} \\ 0 & M_{32} - \frac{M_{31}}{M_{11}} M_{32} & M_{33} - \frac{M_{31}}{M_{11}} M_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{M_{21}}{M_{11}} b_1 \\ b_3 - \frac{M_{31}}{M_{11}} b_1 \end{bmatrix}. \tag{2.5}$$

Here the M_{11} element is called the pivot . Each of the M_{j1}/M_{11} elements is called a multiplier . This operation can be written as

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22}^{(2)} & M_{23}^{(3)} \\ 0 & M_{32}^{(2)} & M_{33}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(2)} \end{bmatrix}. \tag{2.6}$$

Using $M_{22}^{(2)}$ as the pivot this time, we form

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22}^{(2)} & M_{23}^{(3)} \\ 0 & 0 & M_{33}^{(3)} - \frac{M_{32}^{(2)}}{M_{22}^{(2)}} M_{23}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{M_{21}}{M_{11}} b_1 \\ b_3 - \frac{M_{31}}{M_{11}} b_1 - \frac{M_{32}^{(2)}}{M_{22}^{(2)}} b_2^{(2)} \end{bmatrix}. \tag{2.7}$$

2.2 LU decomposition

Through Gaussian elimination, we have transformed the system from

$$Mx = b \tag{2.8}$$

to

$$Ux = y. \tag{2.9}$$

Writing out our Gaussian transformation in the form $U\bar{x} = b$ we have

$$U\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{M_{21}}{M_{11}} & 1 & 0 \\ \frac{M_{32}^{(2)}}{M_{22}^{(2)}} \frac{M_{21}}{M_{11}} - \frac{M_{31}}{M_{11}} & -\frac{M_{32}^{(2)}}{M_{22}^{(2)}} & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2.10}$$

We can verify that the operation matrix K^{-1}, where $K^{-1}U = M$ that takes us to this form is

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{M_{21}}{M_{11}} & 1 & 0 \\ \frac{M_{31}}{M_{11}} & \frac{M_{32}^{(2)}}{M_{22}^{(2)}} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \bar{x} = \bar{b} \tag{2.11}$$

Using this LU decomposition is generally superior to standard Gaussian elimination, since we can use this for many different $\bar{b}$ vectors using the same amount of work.

Our steps are

$$\begin{aligned} b &= Mx \\ &= L\left(Ux\right) \\ &\equiv Ly. \end{aligned} \tag{2.12}$$

We can now solve $Ly = b$, using substitution for y_1, then y_2, and finally y_3. This is called forward substitution .

Finally, we can now solve

$$Ux = y, \tag{2.13}$$

using back substitution .

Note that we produced the vector y as a side effect of performing the Gaussian elimination process.