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Sum of digits of small powers of nine

In a previous post I wondered how to prove that for integer $d \in [1, N]$

$$((N-1)d) \bmod N + ((N-1)d) \operatorname{div} N = N - 1. \tag{1.1}$$

Here's a proof in two steps. First for $N = 10$, and then by search and replace for arbitrary N.

$N = 10$ Let

$$x = 9d = 10a + b, \tag{1.2}$$

where $1 \le a, b < 9$, and let

$$y = a + b, \tag{1.3}$$

the sum of the digits in a base 10 numeral system.
We wish to solve the following integer system of equations

$$\begin{aligned} 9d &= 10a + b \\ y &= a + b \end{aligned}. \tag{1.4}$$

Scaling and subtracting we have

$$10y - 9d = 9b, \tag{1.5}$$

or

$$y = \frac{9}{10}(b + d). \tag{1.6}$$

Because y is an integer, we have to conclude that $b + d$ is a power of 10, and $b + d \ge 10$. Because we have a constraint on the maximum value of this sum

$$b + d \le 2(9), \tag{1.7}$$

we can only conclude that

$$b + d = 10. \tag{1.8}$$

or

$$\boxed{b = 10 - d.} \tag{1.9}$$

Back substitution into eq. (1.2) we have

$$\begin{aligned} 10a &= 9d - b \\ &= 9d - 10 + d \\ &= 10d - 10 \\ &= 10\left(d - 1\right), \end{aligned} \tag{1.10}$$

or

$$\boxed{a = d - 1.} \tag{1.11}$$

Summing eq. (1.11) and eq. (1.9), the sum of digits is

$$a + b = d - 1 + 10 - d = 9. \tag{1.12}$$

For arbitrary N There was really nothing special about $9, 10$ in the above proof, so generalizing requires nothing more than some search and replace. I used the following vim commands for this "proof generalization"

```
:,/For arb/-1 y
:+/For arb/+1
:p
:,$ s/\<9\>/(N-1)/cg
:,$ s/\<10\>/N/cg
:,$ s/numberGame:/&2:/g
```

Let

$$x = (N-1)d = Na + b, \tag{1.13}$$

where $1 \le a, b < N - 1$, and let

$$y = a + b, \tag{1.14}$$

the sum of the digits in a base N numeral system.
We wish to solve the following integer system of equations

$$\begin{aligned} (N-1)d &= Na + b \\ y &= a + b \end{aligned}. \tag{1.15}$$

Scaling and subtracting we have

$$Ny - (N-1)d = (N-1)b, \tag{1.16}$$

or

$$y = \frac{N-1}{N}\left(b+d\right). \tag{1.17}$$

Because y is an integer, we have to conclude that $b+d$ is a power of N, and $b+d \geq N$. Because we have a constraint on the maximum value of this sum

$$b+d \leq 2(N-1), \tag{1.18}$$

we can only conclude that

$$b+d = N. \tag{1.19}$$

or

$$\boxed{b = N-d.} \tag{1.20}$$

Back substitution into eq. (1.13) we have

$$\begin{aligned} Na &= (N-1)d - b \\ &= (N-1)d - N + d \\ &= Nd - N \\ &= N\left(d-1\right), \end{aligned} \tag{1.21}$$

or

$$\boxed{a = d-1.} \tag{1.22}$$

Summing eq. (1.22) and eq. (1.20), the sum of digits is

$$a+b = d-1+N-d = N-1. \tag{1.23}$$

This completes the proof of eq. (1.1).