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Stability of discretized linear differential equations

In class today was a highlight of stability methods for linear multistep methods. To motivate the methods used, it is helpful to take a step back and review stability concepts for LDE systems.

By way of example, consider a second order LDE homogeneous system defined by

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2 = 0. \tag{1.1}$$

Such a system can be solved by assuming an exponential solution, say

$$x(t) = e^{st}. \tag{1.2}$$

Substitution gives

$$e^{st}\left(s^2 + 3s + 2\right) = 0, \tag{1.3}$$

The polynomial part of this equation, the characteristic equation has roots $s = -2, -1$.

The general solution of eq. (1.1) is formed by a superposition of solutions for each value of s

$$x(t) = ae^{-2t} + be^{-t}. \tag{1.4}$$

Independent of any selection of the superposition constants a, b, this function will not blow up as $t \to \infty$.

This "stability" is due to the fact that both of the characteristic equation roots lie in the left hand Argand plane.

Now consider a discretized form of this LDE. This will have the form

$$\begin{aligned}
0 &= \frac{1}{(\Delta t)^2}\left(x_{n+2} - 2x_{n-1} + x_n\right) + \frac{3}{\Delta t}\left(x_{n+1} - x_n\right) + 2x_n \\
&= x_{n+2}\left(\frac{1}{(\Delta t)^2}\right) + x_{n+1}\left(\frac{3}{\Delta t} - \frac{2}{(\Delta t)^2}\right) + x_n\left(\frac{1}{(\Delta t)^2} - \frac{3}{\Delta t} + 2\right),
\end{aligned} \tag{1.5}$$

or

$$0 = x_{n+2} + x_{n+1}\left(3\Delta t - 2\right) + x_n\left(1 - 3\Delta t + 2\left(\Delta t\right)^2\right). \tag{1.6}$$

Note that after discretization, each subsequent index corresponds to a time shift. Also observe that the coefficients of this discretized equation are dependent on the discretization interval size Δt. If the specifics of those coefficients are ignored, a general form with the following structure can be observed

$$0 = x_{n+2}\gamma_0 + x_{n+1}\gamma_1 + x_n\gamma_2. \tag{1.7}$$

It turns out that, much like the LDE solution by characteristic polynomial, it is possible to attack this problem by assuming a solution of the form

$$x_n = Cz^n. \tag{1.8}$$

A time shift index change $x_n \rightarrow x_{n+1}$ results in a power adjustment in this assumed solution. This substitution applied to eq. (1.7) yields

$$0 = Cz^n \left(z^2\gamma_0 + z\gamma_1 + 1\gamma_2 \right), \tag{1.9}$$

Suppose that this polynomial has roots $z \in \{z_1, z_2\}$. A superposition, such as

$$x_n = az_1^n + bz_2^n, \tag{1.10}$$

will also be a solution since insertion of this into the RHS of eq. (1.7) yields

$$az_1^n \left(z_1^2\gamma_0 + z_1\gamma_1 + \gamma_2 \right) + bz_2^n \left(z_2^2\gamma_0 + z_2\gamma_1 + \gamma_2 \right) = az_1^n \times 0 + bz_2^n \times 0. \tag{1.11}$$

The zero equality follows since z_1, z_2 are both roots of the characteristic equation for this discretized LDE. In the discrete z domain stability requires that the roots satisfy the bound $|z| < 1$, a different stability criteria than in the continuous domain. In fact, there is no a-priori guarantee that stability in the continuous domain will imply stability in the discretized domain.

Let's plot those z-domain roots for this example LDE, using $\Delta t \in \{1/2, 1, 2\}$. The respective characteristic polynomials are

$$0 = z^2 - \frac{1}{2}z = z\left(z - \frac{1}{2}\right) \tag{1.12a}$$

$$0 = z^2 + z = z\,(z+1) \tag{1.12b}$$

$$0 = z^2 + 4z + 3 = (z+3)(z+1). \tag{1.12c}$$

These have respective roots

$$z = 0, \frac{1}{2} \tag{1.13a}$$

$$z = 0, -1 \tag{1.13b}$$

$$z = -1, -3 \tag{1.13c}$$

Only the first discretization of these three yields stable solutions in the z domain, although it appears that $\Delta t = 1$ is right on the boundary.