

Stability of discretized linear differential equations

In class today was a highlight of stability methods for linear multistep methods. To motivate the methods used, it is helpful to take a step back and review stability concepts for LDE systems.

By way of example, consider a second order LDE homogeneous system defined by

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2 = 0. \quad (1.1)$$

Such a system can be solved by assuming an exponential solution, say

$$x(t) = e^{st}. \quad (1.2)$$

Substitution gives

$$e^{st} (s^2 + 3s + 2) = 0, \quad (1.3)$$

The polynomial part of this equation, the characteristic equation has roots $s = -2, -1$.

The general solution of eq. (1.1) is formed by a superposition of solutions for each value of s

$$x(t) = ae^{-2t} + be^{-t}. \quad (1.4)$$

Independent of any selection of the superposition constants a, b , this function will not blow up as $t \rightarrow \infty$.

This “stability” is due to the fact that both of the characteristic equation roots lie in the left hand Argand plane.

Now consider a discretized form of this LDE. This will have the form

$$\begin{aligned} 0 &= \frac{1}{(\Delta t)^2} (x_{n+2} - 2x_{n+1} + x_n) + \frac{3}{\Delta t} (x_{n+1} - x_n) + 2x_n \\ &= x_{n+2} \left(\frac{1}{(\Delta t)^2} \right) + x_{n+1} \left(\frac{3}{\Delta t} - \frac{2}{(\Delta t)^2} \right) + x_n \left(\frac{1}{(\Delta t)^2} - \frac{3}{\Delta t} + 2 \right), \end{aligned} \quad (1.5)$$

or

$$0 = x_{n+2} + x_{n+1} (3\Delta t - 2) + x_n (1 - 3\Delta t + 2(\Delta t)^2). \quad (1.6)$$

Note that after discretization, each subsequent index corresponds to a time shift. Also observe that the coefficients of this discretized equation are dependent on the discretization interval size Δt . If the specifics of those coefficients are ignored, a general form with the following structure can be observed

$$0 = x_{n+2}\gamma_0 + x_{n+1}\gamma_1 + x_n\gamma_2. \quad (1.7)$$

It turns out that, much like the LDE solution by characteristic polynomial, it is possible to attack this problem by assuming a solution of the form

$$x_n = Cz^n. \quad (1.8)$$

A time shift index change $x_n \rightarrow x_{n+1}$ results in a power adjustment in this assumed solution. This substitution applied to eq. (1.7) yields

$$0 = Cz^n (z^2\gamma_0 + z\gamma_1 + 1\gamma_2), \quad (1.9)$$

Suppose that this polynomial has roots $z \in \{z_1, z_2\}$. A superposition, such as

$$x_n = az_1^n + bz_2^n, \quad (1.10)$$

will also be a solution since insertion of this into the RHS of eq. (1.7) yields

$$az_1^n (z_1^2\gamma_0 + z_1\gamma_1 + \gamma_2) + bz_2^n (z_2^2\gamma_0 + z_2\gamma_1 + \gamma_2) = az_1^n \times 0 + bz_2^n \times 0. \quad (1.11)$$

The zero equality follows since z_1, z_2 are both roots of the characteristic equation for this discretized LDE. In the discrete z domain stability requires that the roots satisfy the bound $|z| < 1$, a different stability criteria than in the continuous domain. In fact, there is no a-priori guarantee that stability in the continuous domain will imply stability in the discretized domain.

Let's plot those z -domain roots for this example LDE, using $\Delta t \in \{1/2, 1, 2\}$. The respective characteristic polynomials are

$$0 = z^2 - \frac{1}{2}z = z \left(z - \frac{1}{2} \right) \quad (1.12a)$$

$$0 = z^2 + z = z(z + 1) \quad (1.12b)$$

$$0 = z^2 + 4z + 3 = (z + 3)(z + 1). \quad (1.12c)$$

These have respective roots

$$z = 0, \frac{1}{2} \quad (1.13a)$$

$$z = 0, -1 \quad (1.13b)$$

$$z = -1, -3 \quad (1.13c)$$

Only the first discretization of these three yields stable solutions in the z domain, although it appears that $\Delta t = 1$ is right on the boundary.