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2D SHO xy perturbation

Exercise 1.1 **2D SHO xy perturbation. ([1] *pr. 5.4*)**

Given a 2D SHO with Hamiltonian

$$H_0 = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right), \tag{1.1}$$

1. What are the energies and degeneracies of the three lowest states?
2. With perturbation

$$V = m\omega^2 xy, \tag{1.2}$$

calculate the first order energy perturbations and the zeroth order perturbed states.

3. Solve the $H_0 + \delta V$ problem exactly, and compare.

Answer for Exercise 1.1

Part 1. Recall that we have

$$H\left|n_1, n_2\right\rangle = \hbar\omega\left(n_1 + n_2 + 1\right)\left|n_1, n_2\right\rangle, \tag{1.3}$$

So the three lowest energy states are $\left|0,0\right\rangle, \left|1,0\right\rangle, \left|0,1\right\rangle$ with energies $\hbar\omega, 2\hbar\omega, 2\hbar\omega$ respectively (with a two fold degeneracy for the second two energy eigenkets).

Part 2. Consider the action of xy on the $\beta = \left\{\left|0,0\right\rangle, \left|1,0\right\rangle, \left|0,1\right\rangle\right\}$ subspace. Those are

$$\begin{aligned}
xy\left|0,0\right\rangle &= \frac{x_0^2}{2}\left(a + a^\dagger\right)\left(b + b^\dagger\right)\left|0,0\right\rangle \\
&= \frac{x_0^2}{2}\left(b + b^\dagger\right)\left|1,0\right\rangle \\
&= \frac{x_0^2}{2}\left|1,1\right\rangle.
\end{aligned} \tag{1.4}$$

$$
\begin{aligned}
xy\,|1,0\rangle &= \frac{x_0^2}{2}\left(a+a^\dagger\right)\left(b+b^\dagger\right)|1,0\rangle \\
&= \frac{x_0^2}{2}\left(a+a^\dagger\right)|1,1\rangle \\
&= \frac{x_0^2}{2}\left(|0,1\rangle+\sqrt{2}\,|2,1\rangle\right).
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
xy\,|0,1\rangle &= \frac{x_0^2}{2}\left(a+a^\dagger\right)\left(b+b^\dagger\right)|0,1\rangle \\
&= \frac{x_0^2}{2}\left(b+b^\dagger\right)|1,1\rangle \\
&= \frac{x_0^2}{2}\left(|1,0\rangle+\sqrt{2}\,|1,2\rangle\right).
\end{aligned} \tag{1.6}
$$

The matrix representation of $m\omega^2 xy$ with respect to the subspace spanned by basis β above is

$$
xy \sim \frac{1}{2}\hbar\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{1.7}
$$

This diagonalizes with

$$
U = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{U} \end{bmatrix} \tag{1.8a}
$$

$$
\tilde{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{1.8b}
$$

$$
D = \frac{1}{2}\hbar\omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.8c}
$$

$$
xy = UDU^\dagger = UDU. \tag{1.8d}
$$

The unperturbed Hamiltonian in the original basis is

$$
H_0 = \hbar\omega \begin{bmatrix} 1 & 0 \\ 0 & 2I \end{bmatrix}, \tag{1.9}
$$

So the transformation to the diagonal xy basis leaves the initial Hamiltonian unaltered

$$
\begin{aligned}
H_0' &= U^\dagger H_0 U \\
&= \hbar\omega \begin{bmatrix} 1 & 0 \\ 0 & \tilde{U}2I\tilde{U} \end{bmatrix} \\
&= \hbar\omega \begin{bmatrix} 1 & 0 \\ 0 & 2I \end{bmatrix}.
\end{aligned} \tag{1.10}
$$

Now we can compute the first order energy shifts almost by inspection. Writing the new basis as $\beta' = \{|0\rangle, |1\rangle, |2\rangle\}$ those energy shifts are just the diagonal elements from the xy operators matrix representation

$$\begin{aligned} E_0^{(1)} &= \langle 0| V |0\rangle = 0 \\ E_1^{(1)} &= \langle 1| V |1\rangle = \frac{1}{2}\hbar\omega \\ E_2^{(1)} &= \langle 2| V |2\rangle = -\frac{1}{2}\hbar\omega. \end{aligned} \tag{1.11}$$

The new energies are

$$\begin{aligned} &E_0 \rightarrow \hbar\omega \\ &E_1 \rightarrow \hbar\omega\left(2 + \delta/2\right) \\ &E_2 \rightarrow \hbar\omega\left(2 - \delta/2\right). \end{aligned} \tag{1.12}$$

Part 3. For the exact solution, it's possible to rotate the coordinate system in a way that kills the explicit xy term of the perturbation. That we could do this for x, y operators wasn't obvious to me, but after doing so (and rotating the momentum operators the same way) the new operators still have the required commutators. Let

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}. \end{aligned} \tag{1.13}$$

Similarly, for the momentum operators, let

$$\begin{aligned} \begin{bmatrix} p_u \\ p_v \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} \\ &= \begin{bmatrix} p_x\cos\theta + p_y\sin\theta \\ -p_x\sin\theta + p_y\cos\theta \end{bmatrix}. \end{aligned} \tag{1.14}$$

For the commutators of the new operators we have

$$\begin{aligned} \left[u, p_u\right] &= \left[x\cos\theta + y\sin\theta, p_x\cos\theta + p_y\sin\theta\right] \\ &= \left[x, p_x\right]\cos^2\theta + \left[y, p_y\right]\sin^2\theta \\ &= i\hbar\left(\cos^2\theta + \sin^2\theta\right) \\ &= i\hbar. \end{aligned} \tag{1.15}$$

$$\begin{aligned} \left[v, p_v\right] &= \left[-x\sin\theta + y\cos\theta, -p_x\sin\theta + p_y\cos\theta\right] \\ &= \left[x, p_x\right]\sin^2\theta + \left[y, p_y\right]\cos^2\theta \\ &= i\hbar. \end{aligned} \tag{1.16}$$

$$\begin{aligned}
[u, p_v] &= [x\cos\theta + y\sin\theta, -p_x\sin\theta + p_y\cos\theta] \\
&= \cos\theta\sin\theta\left(-[x, p_x] + [y, p_p]\right) \\
&= 0.
\end{aligned} \tag{1.17}$$

$$\begin{aligned}
[v, p_u] &= [-x\sin\theta + y\cos\theta, p_x\cos\theta + p_y\sin\theta] \\
&= \cos\theta\sin\theta\left(-[x, p_x] + [y, p_p]\right) \\
&= 0.
\end{aligned} \tag{1.18}$$

We see that the new operators are canonical conjugate as required. For this problem, we just want a 45 degree rotation, with

$$\begin{aligned}
x &= \frac{1}{\sqrt{2}}(u + v) \\
y &= \frac{1}{\sqrt{2}}(u - v).
\end{aligned} \tag{1.19}$$

We have

$$\begin{aligned}
x^2 + y^2 &= \frac{1}{2}\left((u + v)^2 + (u - v)^2\right) \\
&= \frac{1}{2}\left(2u^2 + 2v^2 + 2uv - 2uv\right) \\
&= u^2 + v^2,
\end{aligned} \tag{1.20}$$

$$\begin{aligned}
p_x^2 + p_y^2 &= \frac{1}{2}\left((p_u + p_v)^2 + (p_u - p_v)^2\right) \\
&= \frac{1}{2}\left(2p_u^2 + 2p_v^2 + 2p_u p_v - 2p_u p_v\right) \\
&= p_u^2 + p_v^2,
\end{aligned} \tag{1.21}$$

and

$$\begin{aligned}
xy &= \frac{1}{2}\left((u + v)(u - v)\right) \\
&= \frac{1}{2}\left(u^2 - v^2\right).
\end{aligned} \tag{1.22}$$

The perturbed Hamiltonian is

$$\begin{aligned}
H_0 + \delta V &= \frac{1}{2m}\left(p_u^2 + p_v^2\right) + \frac{1}{2}m\omega^2\left(u^2 + v^2 + \delta u^2 - \delta v^2\right) \\
&= \frac{1}{2m}\left(p_u^2 + p_v^2\right) + \frac{1}{2}m\omega^2\left(u^2(1 + \delta) + v^2(1 - \delta)\right).
\end{aligned} \tag{1.23}$$

In this coordinate system, the corresponding eigensystem is

$$H\left|n_1, n_2\right\rangle = \hbar\omega\left(1 + n_1\sqrt{1+\delta} + n_2\sqrt{1-\delta}\right)\left|n_1, n_2\right\rangle. \tag{1.24}$$

For small δ

$$n_1\sqrt{1+\delta} + n_2\sqrt{1-\delta} \approx n_1 + n_2 + \frac{1}{2}n_1\delta - \frac{1}{2}n_2\delta, \tag{1.25}$$

so

$$H\left|n_1, n_2\right\rangle \approx \hbar\omega\left(1 + n_1 + n_2 + \frac{1}{2}n_1\delta - \frac{1}{2}n_2\delta\right)\left|n_1, n_2\right\rangle. \tag{1.26}$$

The lowest order perturbed energy levels are

$$\left|0, 0\right\rangle \rightarrow \hbar\omega \tag{1.27}$$

$$\left|1, 0\right\rangle \rightarrow \hbar\omega\left(2 + \frac{1}{2}\delta\right) \tag{1.28}$$

$$\left|0, 1\right\rangle \rightarrow \hbar\omega\left(2 - \frac{1}{2}\delta\right) \tag{1.29}$$

The degeneracy of the $\left|0, 1\right\rangle, \left|1, 0\right\rangle$ states has been split, and to first order match the zeroth order perturbation result.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1