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L_z and $\mathbf{L}^2$ eigenvalues and probabilities for a wave function

Q: [1] 3.17 Given a wave function

$$\psi(r, \theta, \phi) = f(r) \left(x + y + 3z \right), \tag{1.1}$$

(a) Determine if this wave function is an eigenfunction of $\mathbf{L}^2$, and the value of l if it is an eigenfunction.

(b) Determine the probabilities for the particle to be found in any given $|l, m\rangle$ state,

(c) If it is known that ψ is an energy eigenfunction with energy E indicate how we can find $V(r)$.

A: (a) Using

$$\mathbf{L}^2 = -\hbar^2 \left(\frac{1}{\sin^2\theta} \partial_{\phi\phi} + \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \partial_\theta \right) \right), \tag{1.2}$$

and

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{1.3}$$

it's a quick computation to show that

$$\mathbf{L}^2 \psi = 2\hbar^2 \psi = 1(1+1)\hbar^2 \psi, \tag{1.4}$$

so this function is an eigenket of $\mathbf{L}^2$ with an eigenvalue of $2\hbar^2$, which corresponds to $l = 1$, a p-orbital state.

(b) Recall that the angular representation of L_z is

$$L_z = -i\hbar \frac{\partial,}{\partial\phi} \tag{1.5}$$

so we have

$$
\begin{aligned}
L_z x &= i\hbar y \\
L_z y &= -i\hbar x \\
L_z z &= 0,
\end{aligned}
\tag{1.6}
$$

The L_z action on ψ is

$$
L_z \psi = -i\hbar r f(r) \left(-y + x\right). \tag{1.7}
$$

This wave function is not an eigenket of L_z. Expressed in terms of the L_z basis states $e^{im\phi}$, this wave function is

$$
\begin{aligned}
\psi &= r f(r) \left(\sin\theta \left(\cos\phi + \sin\phi\right) + \cos\theta\right) \\
&= r f(r) \left(\frac{\sin\theta}{2}\left(e^{i\phi}\left(1 + \frac{1}{i}\right) + e^{-i\phi}\left(1 - \frac{1}{i}\right)\right) + \cos\theta\right) \\
&= r f(r) \left(\frac{(1-i)\sin\theta}{2} e^{1i\phi} + \frac{(1+i)\sin\theta}{2} e^{-1i\phi} + \cos\theta e^{0i\phi}\right)
\end{aligned}
\tag{1.8}
$$

Assuming that ψ is normalized, the probabilities for measuring $m = 1, -1, 0$ respectively are

$$
\begin{aligned}
P_{\pm 1} &= 2\pi\rho \left|\frac{1 \mp i}{2}\right|^2 \int_0^\pi \sin\theta d\theta \sin^2\theta \\
&= -2\pi\rho \int_1^{-1} du (1 - u^2) \\
&= 2\pi\rho \left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} \\
&= 2\pi\rho \left(2 - \frac{2}{3}\right) \\
&= \frac{8\pi\rho}{3},
\end{aligned}
\tag{1.9}
$$

and

$$
\begin{aligned}
P_0 &= 2\pi\rho \int_0^\pi \sin\theta\cos\theta \\
&= 0,
\end{aligned}
\tag{1.10}
$$

where

$$
\rho = \int_0^\infty r^4 |f(r)|^2 dr. \tag{1.11}
$$

Because the probabilities must sum to 1, this means the $m = \pm 1$ states are equiprobable with $P_\pm = 1/2$, fixing $\rho = 3/16\pi$, even without knowing $f(r)$.

(c) The operator $r^2\mathbf{p}^2$ can be decomposed into a $\mathbf{L}^2$ component and some other portions, from which we can write

$$\begin{aligned} H\psi &= \left(\frac{\mathbf{p}^2}{2m} + V(r)\right)\psi \\ &= \left(-\frac{\hbar^2}{2m}\left(\partial_{rr} + \frac{2}{r}\partial_r - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\right) + V(r)\right)\psi. \end{aligned} \tag{1.12}$$

(See: [1] eq. 6.21)
In this case where $\mathbf{L}^2\psi = 2\hbar^2\psi$ we can rearrange for $V(r)$

$$\begin{aligned} V(r) &= E + \frac{1}{\psi}\frac{\hbar^2}{2m}\left(\partial_{rr} + \frac{2}{r}\partial_r - \frac{2}{r^2}\right)\psi \\ &= E + \frac{1}{f(r)}\frac{\hbar^2}{2m}\left(\partial_{rr} + \frac{2}{r}\partial_r - \frac{2}{r^2}\right)f(r). \end{aligned} \tag{1.13}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1, 1