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Energy estimate for an absolute value potential

Exercise 1.1 Energy estimate for an absolute value potential. ([1] *pr. 5.21*)

Estimate the lowest eigenvalue λ of the differential equation

$$\frac{d^2}{dx^2}\psi + (\lambda - |x|)\,\psi = 0. \tag{1.1}$$

Using α variation with the trial function

$$\psi = \left\{ \begin{array}{ll} c(\alpha - |x|) & |x| < \alpha \\ 0 & |x| > \alpha \end{array} \right. \tag{1.2}$$

Answer for Exercise 1.1

First rewrite the differential equation in a Hamiltonian like fashion

$$H\psi = -\frac{d^2}{\psi}dx^2 + |x|\psi = \lambda\psi. \tag{1.3}$$

We need the derivatives of the trial distribution. The first derivative is

$$\begin{aligned} \frac{d}{dx}\psi &= -c\frac{d}{dx}|x| \\ &= -c\frac{d}{dx}\left(x\theta(x) - x\theta(-x)\right) \\ &= -c\left(\theta(x) - \theta(-x) + x\delta(x) + x\delta(-x)\right) \\ &= -c\left(\theta(x) - \theta(-x) + 2x\delta(x)\right). \end{aligned} \tag{1.4}$$

The second derivative is

$$\begin{aligned} \frac{d^2}{dx^2}\psi &= -c\frac{d}{dx}\left(\theta(x) - \theta(-x) + 2x\delta(x)\right) \\ &= -c\left(\delta(x) + \delta(-x) + 2\delta(x) + 2x\delta'(x)\right) \\ &= -c\left(4\delta(x) + 2x\frac{-\delta(x)}{x}\right) \\ &= -2c\delta(x). \end{aligned} \tag{1.5}$$

This gives

$$H\psi = -2c\delta(x) + |x|c\,(\alpha - |x|)\,. \tag{1.6}$$

We are now set to compute some of the inner products. The normalization is the simplest

$$\begin{aligned}
\langle\psi|\psi\rangle &= c^2\int_{-\alpha}^{\alpha}(\alpha - |x|)^2 dx \\
&= 2c^2\int_{0}^{\alpha}(x-\alpha)^2 dx \\
&= 2c^2\int_{-\alpha}^{0} u^2 du \\
&= 2c^2\left(-\frac{(-\alpha)^3}{3}\right) \\
&= \frac{2}{3}c^2\alpha^3.
\end{aligned} \tag{1.7}$$

For the energy

$$\begin{aligned}
\langle\psi|H\psi\rangle &= c^2\int dx\,(\alpha - |x|)\,(-2\delta(x) + |x|\,(\alpha - |x|)) \\
&= c^2\left(-2\alpha + \int_{-\alpha}^{\alpha} dx\,(\alpha - |x|)^2\,|x|\right) \\
&= c^2\left(-2\alpha + 2\int_{-\alpha}^{0} duu^2\,(u+\alpha)\right) \\
&= c^2\left(-2\alpha + 2\left(\frac{u^4}{4} + \alpha\frac{u^3}{3}\right)\Bigg|_{-\alpha}^{0}\right) \\
&= c^2\left(-2\alpha - 2\left(\frac{\alpha^4}{4} - \frac{\alpha^4}{3}\right)\right) \\
&= c^2\left(-2\alpha + \frac{1}{6}\alpha^4\right).
\end{aligned} \tag{1.8}$$

The energy estimate is

$$\begin{aligned}
\overline{E} &= \frac{\langle\psi|H\psi\rangle}{\langle\psi|\psi\rangle} \\
&= \frac{-2\alpha + \frac{1}{6}\alpha^4}{\frac{2}{3}\alpha^3} \\
&= -\frac{3}{\alpha^2} + \frac{1}{4}\alpha.
\end{aligned} \tag{1.9}$$

This has its minimum at

$$0 = -\frac{6}{\alpha^3} + \frac{1}{4}, \tag{1.10}$$

or

$$\alpha = 2 \times 3^{1/3}. \tag{1.11}$$

Back subst into the energy gives

$$\begin{aligned} \bar{E} &= -\frac{3}{4 \times 3^{2/3}} + \frac{1}{2} 3^{1/3} \\ &= \frac{3^{4/3}}{4} \\ &\approx 1.08. \end{aligned} \tag{1.12}$$

The problem says the exact answer is 1.019, so the variation gets within 6 %.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1