

## Energy estimate for an absolute value potential

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### Exercise 1.1      Energy estimate for an absolute value potential. ([1] pr. 5.21)

Estimate the lowest eigenvalue  $\lambda$  of the differential equation

$$\frac{d^2}{dx^2}\psi + (\lambda - |x|)\psi = 0. \tag{1.1}$$

Using  $\alpha$  variation with the trial function

$$\psi = \begin{cases} c(\alpha - |x|) & |x| < \alpha \\ 0 & |x| > \alpha \end{cases} \tag{1.2}$$

### Answer for Exercise 1.1

First rewrite the differential equation in a Hamiltonian like fashion

$$H\psi = -\frac{d^2}{dx^2}\psi + |x|\psi = \lambda\psi. \tag{1.3}$$

We need the derivatives of the trial distribution. The first derivative is

$$\begin{aligned} \frac{d}{dx}\psi &= -c \frac{d}{dx}|x| \\ &= -c \frac{d}{dx}(x\theta(x) - x\theta(-x)) \\ &= -c(\theta(x) - \theta(-x) + x\delta(x) + x\delta(-x)) \\ &= -c(\theta(x) - \theta(-x) + 2x\delta(x)). \end{aligned} \tag{1.4}$$

The second derivative is

$$\begin{aligned} \frac{d^2}{dx^2}\psi &= -c \frac{d}{dx}(\theta(x) - \theta(-x) + 2x\delta(x)) \\ &= -c(\delta(x) + \delta(-x) + 2\delta(x) + 2x\delta'(x)) \\ &= -c\left(4\delta(x) + 2x\frac{-\delta(x)}{x}\right) \\ &= -2c\delta(x). \end{aligned} \tag{1.5}$$

This gives

$$H\psi = -2c\delta(x) + |x|c(\alpha - |x|). \quad (1.6)$$

We are now set to compute some of the inner products. The normalization is the simplest

$$\begin{aligned} \langle \psi | \psi \rangle &= c^2 \int_{-\alpha}^{\alpha} (\alpha - |x|)^2 dx \\ &= 2c^2 \int_0^{\alpha} (x - \alpha)^2 dx \\ &= 2c^2 \int_{-\alpha}^0 u^2 du \\ &= 2c^2 \left( -\frac{(-\alpha)^3}{3} \right) \\ &= \frac{2}{3}c^2\alpha^3. \end{aligned} \quad (1.7)$$

For the energy

$$\begin{aligned} \langle \psi | H\psi \rangle &= c^2 \int dx (\alpha - |x|) (-2\delta(x) + |x|(\alpha - |x|)) \\ &= c^2 \left( -2\alpha + \int_{-\alpha}^{\alpha} dx (\alpha - |x|)^2 |x| \right) \\ &= c^2 \left( -2\alpha + 2 \int_{-\alpha}^0 du u^2 (u + \alpha) \right) \\ &= c^2 \left( -2\alpha + 2 \left( \frac{u^4}{4} + \alpha \frac{u^3}{3} \right) \Big|_{-\alpha}^0 \right) \\ &= c^2 \left( -2\alpha - 2 \left( \frac{\alpha^4}{4} - \frac{\alpha^4}{3} \right) \right) \\ &= c^2 \left( -2\alpha + \frac{1}{6}\alpha^4 \right). \end{aligned} \quad (1.8)$$

The energy estimate is

$$\begin{aligned} \bar{E} &= \frac{\langle \psi | H\psi \rangle}{\langle \psi | \psi \rangle} \\ &= \frac{-2\alpha + \frac{1}{6}\alpha^4}{\frac{2}{3}\alpha^3} \\ &= -\frac{3}{\alpha^2} + \frac{1}{4}\alpha. \end{aligned} \quad (1.9)$$

This has its minimum at

$$0 = -\frac{6}{\alpha^3} + \frac{1}{4} \quad (1.10)$$

or

$$\alpha = 2 \times 3^{1/3}. \quad (1.11)$$

Back subst into the energy gives

$$\begin{aligned} \bar{E} &= -\frac{3}{4 \times 3^{2/3}} + \frac{1}{2} 3^{1/3} \\ &= \frac{3^{4/3}}{4} \\ &\approx 1.08. \end{aligned} \quad (1.12)$$

The problem says the exact answer is 1.019, so the variation gets within 6 %.

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## Bibliography

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- [1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. [1.1](#)