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Commutators of angular momentum and a central force Hamiltonian

Commutators for angular momentum In problem 1.17 of [2] we are to show that non-commuting operators that both commute with the Hamiltonian, have, in general, degenerate energy eigenvalues. That is

$$[A, H] = [B, H] = 0, \tag{1.1}$$

but

$$[A, B] \neq 0. \tag{1.2}$$

Angular momentum for central force Hamiltonian The problem suggests considering L_x, L_z and a central force Hamiltonian $H = \mathbf{p}^2/2m + V(r)$ as examples.

Let's start with demonstrate these commutators act as expected in these cases.

With $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, we have

$$\begin{aligned} L_x &= y p_z - z p_y \\ L_y &= z p_x - x p_z \\ L_z &= x p_y - y p_x. \end{aligned} \tag{1.3}$$

The L_x, L_z commutator is

$$\begin{aligned} [L_x, L_z] &= \left[y p_z - z p_y, x p_y - y p_x \right] \\ &= \left[y p_z, x p_y \right] - \left[y p_z, y p_x \right] - \left[z p_y, x p_y \right] + \left[z p_y, y p_x \right] \\ &= x p_z \left[y, p_y \right] + z p_x \left[p_y, y \right] \\ &= i\hbar \left(x p_z - z p_x \right) \\ &= -i\hbar L_y \end{aligned} \tag{1.4}$$

cyclicly permuting the indexes shows that no pairs of different $\mathbf{L}$ components commute. For L_y, L_x that is

$$
\begin{aligned}
\left[L_y, L_x\right] &= \left[zp_x - xp_z, yp_z - zp_y\right] \\
&= \left[zp_x, yp_z\right] - \left[zp_x, zp_y\right] - \left[xp_z, yp_z\right] + \left[xp_z, zp_y\right] \\
&= yp_x \left[z, p_z\right] + xp_y \left[p_z, z\right] \\
&= i\hbar \left(yp_x - xp_y\right) \\
&= -i\hbar L_z,
\end{aligned}
\tag{1.5}
$$

and for L_z, L_y

$$
\begin{aligned}
\left[L_z, L_y\right] &= \left[xp_y - yp_x, zp_x - xp_z\right] \\
&= \left[xp_y, zp_x\right] - \left[xp_y, xp_z\right] - \left[yp_x, zp_x\right] + \left[yp_x, xp_z\right] \\
&= zp_y \left[x, p_x\right] + yp_z \left[p_x, x\right] \\
&= i\hbar \left(zp_y - yp_z\right) \\
&= -i\hbar L_x.
\end{aligned}
\tag{1.6}
$$

If these angular momentum components are also shown to commute with themselves (which they do), the commutator relations above can be summarized as

$$
\left[L_a, L_b\right] = i\hbar \epsilon_{abc} L_c.
\tag{1.7}
$$

In the example to consider, we'll have to consider the commutators with $\mathbf{p}^2$ and $V(r)$. Picking any one component of $\mathbf{L}$ is sufficent due to the symmetries of the problem. For example

$$
\begin{aligned}
\left[L_x, \mathbf{p}^2\right] &= \left[yp_z - zp_y, p_x^2 + p_y^2 + p_z^2\right] \\
&= \left[yp_z, \cancel{p_x^2} + p_y^2 + \cancel{p_z^2}\right] - \left[zp_y, \cancel{p_x^2} + \cancel{p_y^2} + p_z^2\right] \\
&= p_z \left[y, p_y^2\right] - p_y \left[z, p_z^2\right] \\
&= p_z 2i\hbar p_y - p_y 2i\hbar p_z \\
&= 0.
\end{aligned}
\tag{1.8}
$$

How about the commutator of $\mathbf{L}$ with the potential? It is sufficient to consider one component again, for example

$$
\begin{aligned}
\left[L_x, V\right] &= \left[yp_z - zp_y, V\right] \\
&= y \left[p_z, V\right] - z \left[p_y, V\right] \\
&= -i\hbar y \frac{\partial V(r)}{\partial z} + i\hbar z \frac{\partial V(r)}{\partial y} \\
&= -i\hbar y \frac{\partial V}{\partial r} \frac{\partial r}{\partial z} + i\hbar z \frac{\partial V}{\partial r} \frac{\partial r}{\partial y} \\
&= -i\hbar y \frac{\partial V}{\partial r} \frac{z}{r} + i\hbar z \frac{\partial V}{\partial r} \frac{y}{r} \\
&= 0.
\end{aligned}
\tag{1.9}
$$

This has shown that all the components of **L** commute with a central force Hamiltonian, and each different component of **L** do not commute. It does not demonstrate the degeneracy, but I do recall that exists for this system.

Matrix example of non-commuting commutators I thought perhaps the problem at hand would be easier if I were to construct some example matrices representing operators that did not commute, but did commuted with a Hamiltonian. I came up with

$$
\begin{aligned}
A &= \begin{bmatrix} \sigma_z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
B &= \begin{bmatrix} \sigma_x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
H &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{1.10}
$$

This system has $[A, H] = [B, H] = 0$, and

$$
[A, B] = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.11}
$$

There is one shared eigenvector between all of A, B, H

$$
|3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.12}
$$

The other eigenvectors for A are

$$
\begin{aligned}
|a_1\rangle &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
|a_2\rangle &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\end{aligned} \tag{1.13}
$$

and for B

$$
\begin{aligned}
|b_1\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
|b_2\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},
\end{aligned} \tag{1.14}
$$

This clearly has the degeneracy sought.
Looking to [1], it appears that it is possible to construct an even simpler example. Let

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ H &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{1.15}$$

Here $[A, B] = -A$, and $[A, H] = [B, H] = 0$, but the Hamiltonian isn't interesting at all physically. A less boring example builds on this. Let

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ H &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{1.16}$$

Here $[A, B] \neq 0$, and $[A, H] = [B, H] = 0$. I don't see a way for any exception to be constructed.

The problem The concrete examples above give some intuition for solving the more abstract problem. Suppose that we are working in a basis that simulaneously diagonalizes operator A and the Hamiltonian H. To make life easy consider the simplest case where this basis is also an eigenbasis for the second operator B for all but two of that operators eigenvectors. For such a system let's write

$$\begin{aligned} H \left|1\right\rangle &= \epsilon_1 \left|1\right\rangle \\ H \left|2\right\rangle &= \epsilon_2 \left|2\right\rangle \\ A \left|1\right\rangle &= a_1 \left|1\right\rangle \\ A \left|2\right\rangle &= a_2 \left|2\right\rangle, \end{aligned} \tag{1.17}$$

where $\left|1\right\rangle$, and $\left|2\right\rangle$ are not eigenkets of B. Because B also commutes with H, we must have

$$\begin{aligned} HB \left|1\right\rangle &= H \left|n\right\rangle \left\langle n\right| B \left|1\right\rangle \\ &= \epsilon_n \left|n\right\rangle B_{n1}, \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} BH \left|1\right\rangle &= B\epsilon_1 \left|1\right\rangle \\ &= \epsilon_1 \left|n\right\rangle \left\langle n\right| B \left|1\right\rangle \\ &= \epsilon_1 \left|n\right\rangle B_{n1}. \end{aligned} \tag{1.19}$$

The commutator is

$$[B,H]\left|1\right\rangle = (\epsilon_1 - \epsilon_n)\left|n\right\rangle B_{n1}. \tag{1.20}$$

Similarly

$$[B,H]\left|2\right\rangle = (\epsilon_2 - \epsilon_n)\left|n\right\rangle B_{n2}. \tag{1.21}$$

For those kets $\left|m\right\rangle \in \{\left|3\right\rangle, \left|4\right\rangle, \cdots\}$ that are eigenkets of B, with $B\left|m\right\rangle = b_m\left|m\right\rangle$, we have

$$\begin{aligned}[B,H]\left|m\right\rangle &= B\epsilon_m\left|m\right\rangle - Hb_m\left|m\right\rangle \\ &= b_m\epsilon_m\left|m\right\rangle - \epsilon_m b_m\left|m\right\rangle \\ &= 0.\end{aligned} \tag{1.22}$$

If the commutator is zero, then we require all its matrix elements

$$\begin{aligned}\left\langle 1\right|[B,H]\left|1\right\rangle &= (\epsilon_1 - \epsilon_1)\, B_{11} \\ \left\langle 2\right|[B,H]\left|1\right\rangle &= (\epsilon_1 - \epsilon_2)\, B_{21} \\ \left\langle 1\right|[B,H]\left|2\right\rangle &= (\epsilon_2 - \epsilon_1)\, B_{12} \\ \left\langle 2\right|[B,H]\left|2\right\rangle &= (\epsilon_2 - \epsilon_2)\, B_{22},\end{aligned} \tag{1.23}$$

to be zero. Because of eq. (1.22) only the matrix elements with respect to states $\left|1\right\rangle, \left|2\right\rangle$ need be considered. Two of the matrix elements above are clearly zero, regardless of the values of B_{11}, and B_{22}, and for the other two to be zero, we must either have

- $B_{21} = B_{12} = 0$, or
- $\epsilon_1 = \epsilon_2$.

If the first condition were true we would have

$$\begin{aligned}B\left|1\right\rangle &= \left|n\right\rangle\left\langle n\right|B\left|1\right\rangle \\ &= \left|n\right\rangle B_{n1} \\ &= \left|1\right\rangle B_{11},\end{aligned} \tag{1.24}$$

and $B\left|2\right\rangle = B_{22}\left|2\right\rangle$. This contradicts the requirement that $\left|1\right\rangle, \left|2\right\rangle$ not be eigenkets of B, leaving only the second option. That second option means there must be a degeneracy in the system.

Bibliography

[1] Ronald M. Aarts. *Commuting Matrices*, 2015. URL http://mathworld.wolfram.com/CommutingMatrices.html. [Online; accessed 22-Oct-2015]. 1

[2] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1