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Angular momentum expectation

Exercise 1.1 **Angular momentum expectation values ([1] *pr. 3.18*)**

Compute the expectation values for the first and second powers of the angular momentum operators with respect to states $\left\lvert {lm} \right\rangle$.

Answer for Exercise 1.1

We can write the expectation values for the L_z powers immediately

$$\left\langle {L_z} \right\rangle = m\hbar, \tag{1.1}$$

and

$$\left\langle {L_z^2} \right\rangle = (m\hbar)^2. \tag{1.2}$$

For the x and y components first express the operators in terms of the ladder operators.

$$\begin{aligned} L_+ &= L_x + i L_y \\ L_- &= L_x - i L_y. \end{aligned} \tag{1.3}$$

Rearranging gives

$$\begin{aligned} L_x &= \frac{1}{2} \left(L_+ + L_- \right) \\ L_y &= \frac{1}{2i} \left(L_+ - L_- \right). \end{aligned} \tag{1.4}$$

The first order expectations $\left\langle {L_x} \right\rangle, \left\langle {L_y} \right\rangle$ are both zero since $\left\langle {L_+} \right\rangle = \left\langle {L_-} \right\rangle$. For the second order expectation values we have

$$\begin{aligned} L_x^2 &= \frac{1}{4} \left(L_+ + L_- \right) \left(L_+ + L_- \right) \\ &= \frac{1}{4} \left(L_+ L_+ + L_- L_- + L_+ L_- + L_- L_+ \right) \\ &= \frac{1}{4} \left(L_+ L_+ + L_- L_- + 2 (L_x^2 + L_y^2) \right) \\ &= \frac{1}{4} \left(L_+ L_+ + L_- L_- + 2 (\mathbf{L}^2 - L_z^2) \right), \end{aligned} \tag{1.5}$$

and

$$
\begin{aligned}
L_y^2 &= -\frac{1}{4}\left(L_+ - L_-\right)\left(L_+ - L_-\right) \\
&= -\frac{1}{4}\left(L_+L_+ + L_-L_- - L_+L_- - L_-L_+\right) \\
&= -\frac{1}{4}\left(L_+L_+ + L_-L_- - 2(L_x^2 + L_y^2)\right) \\
&= -\frac{1}{4}\left(L_+L_+ + L_-L_- - 2(\mathbf{L}^2 - L_z^2)\right) .
\end{aligned}
\tag{1.6}
$$

Any expectation value $\langle lm| L_+L_+ |lm\rangle$ or $\langle lm| L_-L_- |lm\rangle$ will be zero, leaving

$$
\begin{aligned}
\left\langle L_x^2 \right\rangle &= \left\langle L_y^2 \right\rangle \\
&= \frac{1}{4}\left\langle 2(\mathbf{L}^2 - L_z^2)\right\rangle \\
&= \frac{1}{2}\left(\hbar^2 l(l+1) - (\hbar m)^2\right) .
\end{aligned}
\tag{1.7}
$$

Observe that we have

$$
\left\langle L_x^2 \right\rangle + \left\langle L_y^2 \right\rangle + \left\langle L_z^2 \right\rangle = \hbar^2 l(l+1) = \left\langle \mathbf{L}^2 \right\rangle ,
\tag{1.8}
$$

which is the quantum mechanical analogue of the classical scalar equation $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1