A curious proof of the Baker-Campbell-Hausdorff formula

Equation (39) of [1] states the Baker-Campbell-Hausdorff formula for two operators $a, b$ that commute with their commutator $[a, b]$

$$e^a e^b = e^{a+b+[a,b]/2}, \quad \text{(1.1)}$$

and provides the outline of an interesting method of proof. That method is to consider the derivative of

$$f(\lambda) = e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)}, \quad \text{(1.2)}$$

That derivative is

$$\frac{df}{d\lambda} = e^{\lambda a} a e^{\lambda b} e^{-\lambda(a+b)} + e^{\lambda a} b e^{\lambda b} e^{-\lambda(a+b)} - e^{\lambda a} b e^{\lambda b} (a + b) e^{-\lambda(a+b)}$$

$$= e^{\lambda a} \left( a e^{\lambda b} + b e^{\lambda b} - e^{\lambda b} (a + b) \right) e^{-\lambda(a+b)}$$

$$= e^{\lambda a} \left( [a, e^{\lambda b}] + [b, e^{\lambda b}] \right) e^{-\lambda(a+b)}$$

$$= e^{\lambda a} \left( [a, e^{\lambda b}] e^{-\lambda(a+b)} \right). \quad \text{(1.3)}$$

The commutator above is proportional to $[a, b]$

$$[a, e^{\lambda b}] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} [a, b^k]$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} k b^{k-1} [a, b]$$

$$= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} b^{k-1} [a, b]$$

$$= \lambda e^{\lambda b} [a, b], \quad \text{(1.4)}$$

so

$$\frac{df}{d\lambda} = \lambda [a, b] f. \quad \text{(1.5)}$$
To get the above, we should also do the induction demonstration for 
\[ [a, b^k] = kb^{k-1} [a, b]. \]
This clearly holds for \( k = 0, 1 \). For any other \( k \) we have
\[
[a, b^{k+1}] = ab^{k+1} - b^{k+1} a \\
= \left( [a, b^k] + b^k a \right) b - b^{k+1} a \\
= kb^{k-1} [a, b] b + b^k ([a, b] + b a) - b^{k+1} a \\
= kb^k [a, b] + b^k [a, b] \\
= (k + 1)b^k [a, b]
\]
(1.6)

Observe that eq. (1.5) is solved by
\[
f = e^{\lambda^2[a, b]/2},
\]
(1.7)
which gives
\[
e^{\lambda^2[a, b]/2} = e^{\lambda a} e^{\lambda b} e^{-\lambda (a+b)}.
\]
(1.8)

Right multiplication by \( e^{\lambda (a+b)} \) which commutes with \( e^{\lambda^2[a, b]/2} \) and setting \( \lambda = 1 \) recovers eq. (1.1) as desired.

What I wonder looking at this, is what thought process led to trying this in the first place? This is not what I would consider an obvious approach to demonstrating this identity.
Bibliography