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A curious proof of the Baker-Campbell-Hausdorff formula

Equation (39) of [1] states the Baker-Campbell-Hausdorff formula for two operators a, b that commute with their commutator $[a, b]$

$$e^a e^b = e^{a+b+[a,b]/2}, \tag{1.1}$$

and provides the outline of an interesting method of proof. That method is to consider the derivative of

$$f(\lambda) = e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)}, \tag{1.2}$$

That derivative is

$$\begin{aligned}
\frac{df}{d\lambda} &= e^{\lambda a} a e^{\lambda b} e^{-\lambda(a+b)} + e^{\lambda a} b e^{\lambda b} e^{-\lambda(a+b)} - e^{\lambda a} b e^{\lambda b} (a+b) e^{-\lambda(a+b)} \\
&= e^{\lambda a} \left(a e^{\lambda b} + b e^{\lambda b} - e^{\lambda b}(a+b) \right) e^{-\lambda(a+b)} \\
&= e^{\lambda a} \left(\left[a, e^{\lambda b} \right] + \cancel{\left[b, e^{\lambda b} \right]} \right) e^{-\lambda(a+b)} \\
&= e^{\lambda a} \left[a, e^{\lambda b} \right] e^{-\lambda(a+b)}.
\end{aligned} \tag{1.3}$$

The commutator above is proportional to $[a, b]$

$$\begin{aligned}
\left[a, e^{\lambda b} \right] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[a, b^k \right] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} k b^{k-1} [a, b] \\
&= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} b^{k-1} [a, b] \\
&= \lambda e^{\lambda b} [a, b],
\end{aligned} \tag{1.4}$$

so

$$\frac{df}{d\lambda} = \lambda [a, b] f. \tag{1.5}$$

To get the above, we should also do the induction demonstration for $\left[a, b^k\right] = kb^{k-1}\left[a, b\right]$. This clearly holds for $k = 0, 1$. For any other k we have

$$\begin{aligned}\left[a, b^{k+1}\right] &= ab^{k+1} - b^{k+1}a \\ &= \left(\left[a, b^k\right] + b^k a\right) b - b^{k+1}a \\ &= kb^{k-1}\left[a, b\right] b + b^k \left(\left[a, b\right] + \cancel{ba}\right) - \cancel{b^{k+1}a} \\ &= kb^k \left[a, b\right] + b^k \left[a, b\right] \\ &= (k+1)b^k \left[a, b\right] \qquad \square\end{aligned} \tag{1.6}$$

Observe that eq. (1.5) is solved by

$$f = e^{\lambda^2 [a,b]/2}, \tag{1.7}$$

which gives

$$e^{\lambda^2 [a,b]/2} = e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)}. \tag{1.8}$$

Right multiplication by $e^{\lambda(a+b)}$ which commutes with $e^{\lambda^2[a,b]/2}$ and setting $\lambda = 1$ recovers eq. (1.1) as desired.

What I wonder looking at this, is what thought process led to trying this in the first place? This is not what I would consider an obvious approach to demonstrating this identity.

Bibliography

[1] Roy J Glauber. Some notes on multiple-boson processes. *Physical Review*, 84(3):395, 1951. 1