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bra-ket manipulation problems

Exercise 1.1 **Some bra-ket manipulation problems. ([1] *pr. 1.4*)**

Using braket logic expand

1.
$$\operatorname{tr} XY \tag{1.1}$$

2.
$$(XY)^\dagger \tag{1.2}$$

3.
$$e^{if(A)}, \tag{1.3}$$

where A is Hermitian with a complete set of eigenvalues.

4.
$$\sum_{a'} \Psi_{a'}(\mathbf{x}')^* \Psi_{a'}(\mathbf{x}''), \tag{1.4}$$

where $\Psi_{a'}(\mathbf{x}'') = \langle \mathbf{x}' | a' \rangle$.

Answer for Exercise 1.1

Part 1.

$$\begin{aligned}
\operatorname{tr} XY &= \sum_a \langle a| XY |a\rangle \\
&= \sum_{a,b} \langle a| X |b\rangle \langle b| Y |a\rangle \\
&= \sum_{a,b} \langle b| Y |a\rangle \langle a| X |b\rangle \\
&= \sum_{a,b} \langle b| YX |b\rangle \\
&= \operatorname{tr} YX.
\end{aligned} \tag{1.5}$$

Part 2.

$$\begin{aligned}
\langle a|\,(XY)^\dagger\,|b\rangle &= (\langle b|\,XY\,|a\rangle)^* \\
&= \sum_c (\langle b|\,X\,|c\rangle\,\langle c|\,Y\,|a\rangle)^* \\
&= \sum_c (\langle b|\,X\,|c\rangle)^*\,(\langle c|\,Y\,|a\rangle)^* \\
&= \sum_c (\langle c|\,Y\,|a\rangle)^*\,(\langle b|\,X\,|c\rangle)^* \\
&= \sum_c \langle a|\,Y^\dagger\,|c\rangle\,\langle c|\,X^\dagger\,|b\rangle \\
&= \langle a|\,Y^\dagger X^\dagger\,|b\rangle\,,
\end{aligned} \tag{1.6}$$

so $(XY)^\dagger = Y^\dagger X^\dagger$.

Part 3. Let's presume that the function f has a Taylor series representation

$$f(A) = \sum_r b_r A^r. \tag{1.7}$$

If the eigenvalues of A are given by

$$A\,|a_s\rangle = a_s\,|a_s\rangle\,, \tag{1.8}$$

this operator can be expanded like

$$\begin{aligned}
A &= \sum_{a_s} A\,|a_s\rangle\,\langle a_s| \\
&= \sum_{a_s} a_s\,|a_s\rangle\,\langle a_s|\,,
\end{aligned} \tag{1.9}$$

To compute powers of this operator, consider first the square

$$\begin{aligned}
A^2 &= \\
&= \sum_{a_s} a_s\,|a_s\rangle\,\langle a_s| \sum_{a_r} a_r\,|a_r\rangle\,\langle a_r| \\
&= \sum_{a_s,a_r} a_s a_r\,|a_s\rangle\,\langle a_s|\,|a_r\rangle\,\langle a_r| \\
&= \sum_{a_s,a_r} a_s a_r\,|a_s\rangle\,\delta_{sr}\,\langle a_r| \\
&= \sum_{a_s} a_s^2\,|a_s\rangle\,\langle a_s|\,.
\end{aligned} \tag{1.10}$$

The pattern for higher powers will clearly just be

$$A^k = \sum_{a_s} a_s^k\,|a_s\rangle\,\langle a_s|\,, \tag{1.11}$$

so the expansion of $f(A)$ will be

$$
\begin{aligned}
f(A) &= \sum_r b_r A^r \\
&= \sum_r b_r \sum_{a_s} a_s^r \left|a_s\right\rangle \left\langle a_s\right| \\
&= \sum_{a_s} \left(\sum_r b_r a_s^r \right) \left|a_s\right\rangle \left\langle a_s\right| \\
&= \sum_{a_s} f(a_s) \left|a_s\right\rangle \left\langle a_s\right| .
\end{aligned} \tag{1.12}
$$

The exponential expansion is

$$
\begin{aligned}
e^{if(A)} &= \sum_t \frac{i^t}{t!} f^t(A) \\
&= \sum_t \frac{i^t}{t!} \left(\sum_{a_s} f(a_s) \left|a_s\right\rangle \left\langle a_s\right| \right)^t \\
&= \sum_t \frac{i^t}{t!} \sum_{a_s} f^t(a_s) \left|a_s\right\rangle \left\langle a_s\right| \\
&= \sum_{a_s} e^{if(a_s)} \left|a_s\right\rangle \left\langle a_s\right| .
\end{aligned} \tag{1.13}
$$

Part 4.

$$
\begin{aligned}
\sum_{a'} \Psi_{a'}(\mathbf{x}')^* \Psi_{a'}(\mathbf{x}'') &= \sum_{a'} \left\langle \mathbf{x}' \middle| a' \right\rangle^* \left\langle \mathbf{x}'' \middle| a' \right\rangle \\
&= \sum_{a'} \left\langle a' \middle| \mathbf{x}' \right\rangle \left\langle \mathbf{x}'' \middle| a' \right\rangle \\
&= \sum_{a'} \left\langle \mathbf{x}'' \middle| a' \right\rangle \left\langle a' \middle| \mathbf{x}' \right\rangle \\
&= \left\langle \mathbf{x}'' \middle| \mathbf{x}' \right\rangle \\
&= \delta_{\mathbf{x}''-\mathbf{x}'} .
\end{aligned} \tag{1.14}
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Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1