Peeter Joot
peeter.joot@gmail.com

Chebychev antenna design

In our text [1] is a design procedure that applies Chebychev polynomials to the selection of current magnitudes for an evenly spaced array of identical antennas placed along the z-axis.

For an even number $2M$ of identical antennas placed at positions $\mathbf{r}_m = (d/2)\,(2m-1)\,\mathbf{e}_3$, the array factor is

$$\mathrm{AF} = \sum_{m=-N}^{N} I_m e^{-jk\hat{\mathbf{r}}\cdot\mathbf{r}_m}. \tag{1.1}$$

Assuming the currents are symmetric $I_{-m} = I_m$, with $\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and $u = \frac{\pi d}{\lambda}\cos\theta$, this is

$$\begin{aligned}
\mathrm{AF} &= \sum_{m=-N}^{N} I_m e^{-jk(d/2)(2m-1)\cos\theta} \\
&= 2\sum_{m=1}^{N} I_m \cos\left(k(d/2)(2m-1)\cos\theta\right) \\
&= 2\sum_{m=1}^{N} I_m \cos\left((2m-1)u\right).
\end{aligned} \tag{1.2}$$

This is a sum of only odd cosines, and can be expanded as a sum that includes all the odd powers of $\cos u$. Suppose for example that this is a four element array with $N = 2$. In this case the array factor has the form

$$\begin{aligned}
\mathrm{AF} &= 2\left(I_1\cos u + I_2\left(4\cos^3 u - 3\cos u\right)\right) \\
&= 2\left((I_1 - 3I_2)\cos u + 4I_2\cos^3 u\right).
\end{aligned} \tag{1.3}$$

The design procedure in the text sets $\cos u = z/z_0$, and then equates this to $T_3(z) = 4z^3 - 3z$ to determine the current amplitudes I_m. That is

$$\frac{2I_1 - 6I_2}{z_0} z + \frac{8I_2}{z_0^3} z^3 = -3z + 4z^3, \tag{1.4}$$

or

$$
\begin{aligned}
\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} 2/z_0 & -6/z_0 \\ 0 & 8/z_0^3 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
&= \frac{z_0}{2} \begin{bmatrix} 3(z_0^2 - 1) \\ z_0^2 \end{bmatrix}.
\end{aligned}
\tag{1.5}
$$

The currents in the array factor are fully determined up to a scale factor, reducing the array factor to

$$
\mathrm{AF} = 4z_0^3 \cos^3 u - 3z_0 \cos u. \tag{1.6}
$$

The zeros of this array factor are located at the zeros of

$$
T_3(z_0 \cos u) = \cos(3 \cos^{-1} (z_0 \cos u)), \tag{1.7}
$$

which are at $3 \cos^{-1} (z_0 \cos u) = \pi/2 + m\pi = \pi \left(m + \frac{1}{2}\right)$

$$
\begin{aligned}
\cos u &= \frac{1}{z_0} \cos \left(\frac{\pi}{3} \left(m + \frac{1}{2} \right) \right) \\
&= \left\{ 0, \pm \frac{\sqrt{3}}{2z_0} \right\}.
\end{aligned}
\tag{1.8}
$$

showing that the scaling factor z_0 effects the locations of the zeros. It also allows the values at the extremes $\cos u = \pm 1$, to increase past the ± 1 non-scaled limit values. These effects can be explored in `http://goo.gl/KPqcjX`, but can also be seen in fig. 1.1.

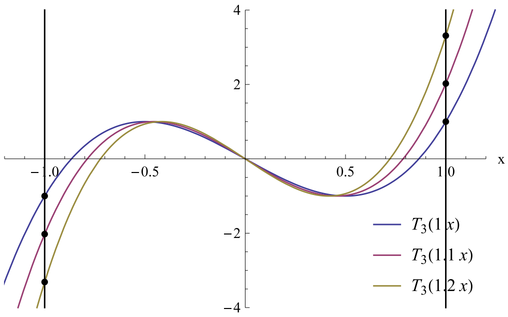

Figure 1.1: $T_3(z_0 x)$ for a few different scale factors z_0.

The scale factor can be fixed for a desired maximum power gain. For RdB, that will be when

$$
20 \log_{10} \cosh(3 \cosh^{-1} z_0) = R\mathrm{dB}, \tag{1.9}
$$

or

$$
z_0 = \cosh \left(\frac{1}{3} \cosh^{-1} \left(10^{\frac{R}{20}} \right) \right). \tag{1.10}
$$

For $R = 30$ dB (say), we have $z_0 = 2.1$, and

$$\text{AF} = 40\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right) - 6.4\cos\left(\frac{\pi d}{\lambda}\cos\theta\right). \tag{1.11}$$

These are plotted in fig. 1.2 (linear scale), and fig. 1.3 (dB scale) for a couple values of d/λ.

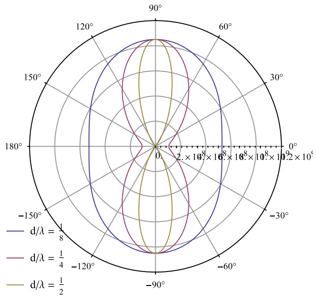

Figure 1.2: T_3 fitting of $N = 4$ array.

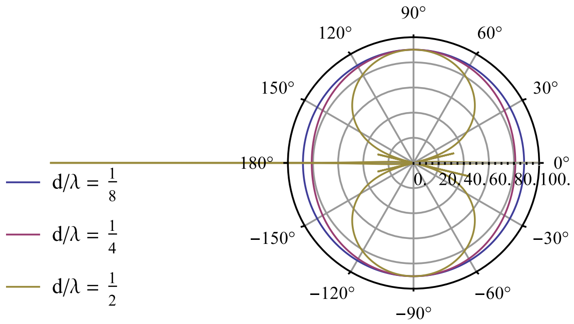

Figure 1.3: T_3 fitting of $N = 4$ array (dB scale).

A Manipulate for exploring the d/λ dependence is available in `http://goo.gl/8FhUwC`.

Bibliography

[1] Constantine A Balanis. *Antenna theory: analysis and design*. John Wiley & Sons, 3rd edition, 2005. 1