Peeter Joot
peeter.joot@gmail.com

Antenna array design with Chebychev polynomials

Prof. Eleftheriades desribed a Chebychev antenna array design method that looks different than the one of the text [1].

Portions of that procedure are like that of the text. For example, if a side lobe level of $20 \log_{10} R$ is desired, a scaling factor

$$x_0 = \cosh\left(\frac{1}{m}\cosh^{-1} R\right), \tag{1.1}$$

is used. Given N elements in the array, a Chebychev polynomial of degree $m = N - 1$ is used. That is

$$T_m(x) = \cos\left(m \cos^{-1} x\right). \tag{1.2}$$

Observe that the roots x'_n of this polynomial lie where

$$m \cos^{-1} x'_n = \frac{\pi}{2} \pm \pi n, \tag{1.3}$$

or

$$x'_n = \cos\left(\frac{\pi}{2m}(2n \pm 1)\right), \tag{1.4}$$

The class notes use the negative sign, and number $n = 1, 2, \cdots, m$. It is noted that the roots are symmetric with $x'_1 = -x'_m$, which can be seen by direct expansion

$$\begin{aligned}
x'_{m-r} &= \cos\left(\frac{\pi}{2m}(2(m-r)-1)\right) \\
&= \cos\left(\pi - \frac{\pi}{2m}(2r+1)\right) \\
&= -\cos\left(\frac{\pi}{2m}(2r+1)\right) \\
&= -\cos\left(\frac{\pi}{2m}(2(r+1)-1)\right) \\
&= -x'_{r+1}. \qquad \square
\end{aligned} \tag{1.5}$$

The next step in the procedure is the identification

$$
\begin{aligned}
u'_n &= 2\cos^{-1}\left(\frac{x'_n}{x_0}\right) \\
z_n &= e^{ju'_n}.
\end{aligned}
\tag{1.6}
$$

This has a factor of two that does not appear in the Balanis design method. It seems plausible that this factor of two was introduced so that the roots of the array factor z_n are conjugate pairs. Since $\cos^{-1}(-z) = \pi - \cos^{-1} z$, this choice leads to such conjugate pairs

$$
\begin{aligned}
\exp\left(ju'_{m-r}\right) &= \exp\left(j2\cos^{-1}\left(\frac{x'_{m-r}}{x_0}\right)\right) \\
&= \exp\left(j2\cos^{-1}\left(-\frac{x'_{r+1}}{x_0}\right)\right) \\
&= \exp\left(j2\left(\pi - \cos^{-1}\left(\frac{x'_{r+1}}{x_0}\right)\right)\right) \\
&= \exp\left(-ju_{r+1}\right).
\end{aligned}
\tag{1.7}
$$

Because of this, the array factor can be written

$$
\begin{aligned}
\mathrm{AF} &= (z - z_1)(z - z_2)\cdots(z - z_{m-1})(z - z_m) \\
&= (z - z_1)(z - z_1^*)(z - z_2)(z - z_2^*)\cdots \\
&= \left(z^2 - z(z_1 + z_1^*) + 1\right)\left(z^2 - z(z_2 + z_2^*) + 1\right)\cdots \\
&= \left(z^2 - 2z\cos\left(2\cos^{-1}\left(\frac{x'_1}{x_0}\right)\right) + 1\right)\left(z^2 - 2z\cos\left(2\cos^{-1}\left(\frac{x'_2}{x_0}\right)\right) + 1\right)\cdots \\
&= \left(z^2 - 2z\left(2\left(\frac{x'_1}{x_0}\right)^2 - 1\right) + 1\right)\left(z^2 - 2z\left(2\left(\frac{x'_2}{x_0}\right)^2 - 1\right) + 1\right)\cdots
\end{aligned}
\tag{1.8}
$$

When m is even, there will only be such conjugate pairs of roots. When m is odd, the remaining factor will be

$$
z - e^{2j\cos^{-1}(0/x_0)} = z - e^{2j\pi/2} = z - e^{j\pi} = z + 1. \tag{1.9}
$$

Example 1.1: Expand AF for $N = 4$.

The roots of $T_3(x)$ are

$$
x'_n \in \left\{0, \pm\frac{\sqrt{3}}{2}\right\}, \tag{1.10}
$$

so the array factor is

$$
\begin{aligned}
\mathrm{AF} &= \left(z^2 + z\left(2 - \frac{3}{x_0^2}\right) + 1\right)(z + 1) \\
&= z^3 + 3z^2\left(1 - \frac{1}{x_0^2}\right) + 3z\left(1 - \frac{1}{x_0^2}\right) + 1.
\end{aligned}
\tag{1.11}
$$

With $20\log_{10} R = 30\text{dB}$, $x_0 = 2.1$, so this is

$$\text{AF} = z^3 + 2.33089z^2 + 2.33089z + 1. \tag{1.12}$$

With

$$\begin{aligned} z &= e^{j(u+u_0)} \\ &= e^{jkd\cos\theta + jku_0}, \end{aligned} \tag{1.13}$$

the array factor takes the form

$$\text{AF} = e^{j3kd\cos\theta + j3ku_0} + 2.33089e^{j2kd\cos\theta + j2ku_0} + 2.33089e^{jkd\cos\theta + jku_0} + 1. \tag{1.14}$$

This array function is highly phase dependent, plotted for $u_0 = 0$ in fig. 1.1, and fig. 1.2.

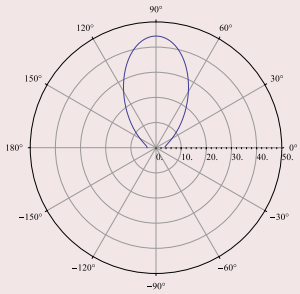

Figure 1.1: Plot with $u_0 = 0, d = \lambda/4$.

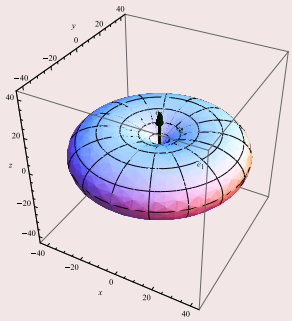

Figure 1.2: Spherical plot with $u_0 = 0, d = \lambda/4$.

This can be directed along a single direction (z-axis) with higher phase choices as illustrated in fig. 1.3, and fig. 1.4.

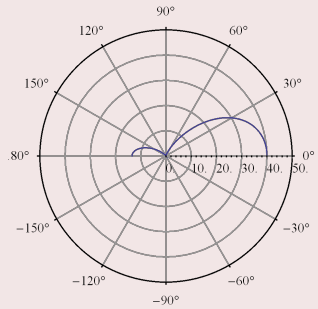

Figure 1.3: Plot with $u_0 = 3.5, d = 0.4\lambda$.

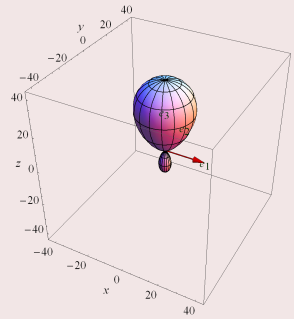

Figure 1.4: Spherical plot with $u_0 = 3.5, d = 0.4\lambda$.

These can be explored interactively in `http://goo.gl/DRDIsr`.

Bibliography

[1] Constantine A Balanis. *Antenna theory: analysis and design*. John Wiley & Sons, 3rd edition, 2005. 1