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Time reversal behavior of solutions to crystal spin Hamiltonian

Exercise 1.1 **Time reversal behavior of solutions to crystal spin Hamiltonian ([1] *pr. 4.12*)**

Solve the spin 1 Hamiltonian

$$H = A S_z^2 + B(S_x^2 - S_y^2). \tag{1.1}$$

Is this Hamiltonian invariant under time reversal?
How do the eigenkets change under time reversal?

Answer for Exercise 1.1

In spinMatrices.nb the matrix representation of the Hamiltonian is found to be

$$H = \hbar^2 \begin{bmatrix} A + \frac{B}{2} & 0 & \frac{B}{2} \\ -\frac{iB}{\sqrt{2}} & B & -\frac{iB}{\sqrt{2}} \\ \frac{B}{2} & 0 & A + \frac{B}{2} \end{bmatrix}. \tag{1.2}$$

The eigenvalues are

$$\left\{ \hbar^2 A, \hbar^2 B, \hbar^2 (A + B) \right\}, \tag{1.3}$$

and the respective eigenvalues (unnormalized) are

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{i\sqrt{2}B}{A} \\ 1 \end{bmatrix} \right\}. \tag{1.4}$$

Under time reversal, the Hamiltonian is

$$H \rightarrow A(-S_z)^2 + B((-S_x)^2 - (-S_y)^2) = H, \tag{1.5}$$

so we expect the eigenkets for this Hamiltonian to vary by at most a phase factor. To check this, first recall that the time reversal action on a spin one state is

$$\Theta \left| 1, m \right\rangle = (-1)^m \left| 1, -m \right\rangle, \tag{1.6}$$

or

$$
\begin{aligned}
\Theta |1\rangle &= - |-1\rangle \\
\Theta |0\rangle &= |0\rangle \\
\Theta |-1\rangle &= - |1\rangle .
\end{aligned}
\tag{1.7}
$$

Let's write the eigenkets respectively as

$$
\begin{aligned}
|A\rangle &= - |1\rangle + |-1\rangle \\
|B\rangle &= |0\rangle \\
|A+B\rangle &= |1\rangle + |-1\rangle - \frac{i\sqrt{2}B}{A} |0\rangle .
\end{aligned}
\tag{1.8}
$$

Noting that the time reversal operator maps complex numbers onto their conjugates, the time reversed eigenkets are

$$
\begin{aligned}
|A\rangle &\rightarrow |-1\rangle - |-1\rangle = - |A\rangle \\
|B\rangle &\rightarrow |0\rangle = |B\rangle \\
|A+B\rangle &\rightarrow - |1\rangle - |-1\rangle + \frac{i\sqrt{2}B}{A} |0\rangle = - |A+B\rangle .
\end{aligned}
\tag{1.9}
$$

Up to a sign, the time reversed states match the unreversed states.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1