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Alternate Dirac equation representation

Exercise 1.1 (*phy1520 2015 midterm pr. 2*)

Given an alternate representation of the Dirac equation

$$H = \begin{bmatrix} mc^2 + V_0 & c\hat{p} \\ c\hat{p} & -mc^2 + V_0 \end{bmatrix}, \tag{1.1}$$

calculate

1. the constant momentum plane wave solutions,
2. the constant momentum hyperbolic solutions,
3. the Heisenberg velocity operator $\hat{v}$, and
4. find the form of the probability density current.

Answer for Exercise 1.1

Part 1. The action of the Hamiltonian on

$$\psi = e^{ikx - iEt/\hbar} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{1.2}$$

is

$$\begin{aligned} H\psi &= \begin{bmatrix} mc^2 + V_0 & c(-i\hbar)ik \\ c(-i\hbar)ik & -mc^2 + V_0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} e^{ikx - iEt/\hbar} \\ &= \begin{bmatrix} mc^2 + V_0 & c\hbar k \\ c\hbar k & -mc^2 + V_0 \end{bmatrix} \psi. \end{aligned} \tag{1.3}$$

Writing

$$H_k = \begin{bmatrix} mc^2 + V_0 & c\hbar k \\ c\hbar k & -mc^2 + V_0 \end{bmatrix} \tag{1.4}$$

the characteristic equation is

$$
\begin{aligned}
0 &= (mc^2 + V_0 - \lambda)(-mc^2 + V_0 - \lambda) - (c\hbar k)^2 \\
&= \left((\lambda - V_0)^2 - (mc^2)^2\right) - (c\hbar k)^2,
\end{aligned}
\tag{1.5}
$$

so

$$
\lambda = V_0 \pm \epsilon,
\tag{1.6}
$$

where

$$
\epsilon^2 = (mc^2)^2 + (c\hbar k)^2.
\tag{1.7}
$$

We've got

$$
\begin{aligned}
H - (V_0 + \epsilon) &= \begin{bmatrix} mc^2 - \epsilon & c\hbar k \\ c\hbar k & -mc^2 - \epsilon \end{bmatrix} \\
H - (V_0 - \epsilon) &= \begin{bmatrix} mc^2 + \epsilon & c\hbar k \\ c\hbar k & -mc^2 + \epsilon \end{bmatrix},
\end{aligned}
\tag{1.8}
$$

so the eigenkets are

$$
\begin{aligned}
\left|V_0 + \epsilon\right\rangle &\propto \begin{bmatrix} -c\hbar k \\ mc^2 - \epsilon \end{bmatrix} \\
\left|V_0 - \epsilon\right\rangle &\propto \begin{bmatrix} -c\hbar k \\ mc^2 + \epsilon \end{bmatrix}.
\end{aligned}
\tag{1.9}
$$

Up to an arbitrary phase for each, these are

$$
\begin{aligned}
\left|V_0 + \epsilon\right\rangle &= \frac{1}{\sqrt{2\epsilon(\epsilon - mc^2)}} \begin{bmatrix} c\hbar k \\ \epsilon - mc^2 \end{bmatrix} \\
\left|V_0 - \epsilon\right\rangle &= \frac{1}{\sqrt{2\epsilon(\epsilon + mc^2)}} \begin{bmatrix} -c\hbar k \\ \epsilon + mc^2 \end{bmatrix}
\end{aligned}
\tag{1.10}
$$

We can now write

$$
H_k = E \begin{bmatrix} V_0 + \epsilon & 0 \\ 0 & V_0 - \epsilon \end{bmatrix} E^{-1},
\tag{1.11}
$$

where

$$
\begin{aligned}
E &= \frac{1}{\sqrt{2\epsilon}} \begin{bmatrix} \frac{c\hbar k}{\sqrt{\epsilon - mc^2}} & -\frac{c\hbar k}{\sqrt{\epsilon + mc^2}} \\ \sqrt{\epsilon - mc^2} & \sqrt{\epsilon + mc^2} \end{bmatrix}, \qquad k > 0 \\
E &= \frac{1}{\sqrt{2\epsilon}} \begin{bmatrix} -\frac{c\hbar k}{\sqrt{\epsilon - mc^2}} & -\frac{c\hbar k}{\sqrt{\epsilon + mc^2}} \\ -\sqrt{\epsilon - mc^2} & \sqrt{\epsilon + mc^2} \end{bmatrix}, \qquad k < 0.
\end{aligned}
\tag{1.12}
$$

Here the signs have been adjusted to ensure the transformation matrix has a unit determinant.

Observe that there's redundancy in this matrix since $c\hbar|k|/\sqrt{\epsilon - mc^2} = \sqrt{\epsilon + mc^2}$, and $c\hbar|k|/\sqrt{\epsilon + mc^2} = \sqrt{\epsilon - mc^2}$, which allows the transformation matrix to be written in the form of a rotation matrix

$$
\begin{aligned}
E &= \frac{1}{\sqrt{2\epsilon}} \begin{bmatrix} \frac{c\hbar k}{\sqrt{\epsilon - mc^2}} & -\frac{c\hbar k}{\sqrt{\epsilon + mc^2}} \\ \frac{c\hbar k}{\sqrt{\epsilon + mc^2}} & \frac{c\hbar k}{\sqrt{\epsilon - mc^2}} \end{bmatrix}, \qquad k > 0 \\
E &= \frac{1}{\sqrt{2\epsilon}} \begin{bmatrix} -\frac{c\hbar k}{\sqrt{\epsilon - mc^2}} & -\frac{c\hbar k}{\sqrt{\epsilon + mc^2}} \\ \frac{c\hbar k}{\sqrt{\epsilon + mc^2}} & -\frac{c\hbar k}{\sqrt{\epsilon - mc^2}} \end{bmatrix}, \qquad k < 0
\end{aligned} \tag{1.13}
$$

With

$$
\begin{aligned}
\cos\theta &= \frac{c\hbar |k|}{\sqrt{2\epsilon(\epsilon - mc^2)}} = \frac{\sqrt{\epsilon + mc^2}}{\sqrt{2\epsilon}} \\
\sin\theta &= \frac{c\hbar k}{\sqrt{2\epsilon(\epsilon + mc^2)}} = \frac{\operatorname{sgn}(k)\sqrt{\epsilon - mc^2}}{\sqrt{2\epsilon}},
\end{aligned} \tag{1.14}
$$

the transformation matrix (and eigenkets) is

$$
\boxed{E = \begin{bmatrix} |V_0 + \epsilon\rangle & |V_0 - \epsilon\rangle \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.} \tag{1.15}
$$

Observe that eq. (1.14) can be simplified by using double angle formulas

$$
\begin{aligned}
\cos(2\theta) &= \frac{\left(\epsilon + mc^2\right)}{2\epsilon} - \frac{\left(\epsilon - mc^2\right)}{2\epsilon} \\
&= \frac{1}{2\epsilon}\left(\epsilon + mc^2 - \epsilon + mc^2\right) \\
&= \frac{mc^2}{\epsilon},
\end{aligned} \tag{1.16}
$$

and

$$
\begin{aligned}
\sin(2\theta) &= 2\frac{1}{2\epsilon}\operatorname{sgn}(k)\sqrt{\epsilon^2 - (mc^2)^2} \\
&= \frac{\hbar k c}{\epsilon}.
\end{aligned} \tag{1.17}
$$

This allows all the θ dependence on $\hbar kc$ and mc^2 to be expressed as a ratio of momenta

$$
\boxed{\tan(2\theta) = \frac{\hbar k}{mc}.} \tag{1.18}
$$

Part 2. For a wave function of the form

$$
\psi = e^{kx - iEt/\hbar} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{1.19}
$$

some of the work above can be recycled if we substitute $k \to -ik$, which yields unnormalized eigenfunctions

$$
\begin{aligned}
|V_0 + \epsilon\rangle &\propto \begin{bmatrix} ic\hbar k \\ mc^2 - \epsilon \end{bmatrix} \\
|V_0 - \epsilon\rangle &\propto \begin{bmatrix} ic\hbar k \\ mc^2 + \epsilon \end{bmatrix},
\end{aligned}
\tag{1.20}
$$

where

$$
\epsilon^2 = (mc^2)^2 - (c\hbar k)^2. \tag{1.21}
$$

The squared magnitude of these wavefunctions are

$$
\begin{aligned}
(c\hbar k)^2 + (mc^2 \mp \epsilon)^2 &= (c\hbar k)^2 + (mc^2)^2 + \epsilon^2 \mp 2mc^2\epsilon \\
&= (c\hbar k)^2 + (mc^2)^2 + (mc^2)^2 \mp (c\hbar k)^2 - 2mc^2\epsilon \\
&= 2(mc^2)^2 \mp 2mc^2\epsilon \\
&= 2mc^2(mc^2 \mp \epsilon),
\end{aligned}
\tag{1.22}
$$

so, up to a constant phase for each, the normalized kets are

$$
\begin{aligned}
|V_0 + \epsilon\rangle &= \frac{1}{\sqrt{2mc^2(mc^2 - \epsilon)}} \begin{bmatrix} ic\hbar k \\ mc^2 - \epsilon \end{bmatrix} \\
|V_0 - \epsilon\rangle &= \frac{1}{\sqrt{2mc^2(mc^2 + \epsilon)}} \begin{bmatrix} ic\hbar k \\ mc^2 + \epsilon \end{bmatrix},
\end{aligned}
\tag{1.23}
$$

After the $k \rightarrow -ik$ substitution, H_k is not Hermitian, so these kets aren't expected to be orthonormal, which is readily verified

$$
\begin{aligned}
\langle V_0 + \epsilon | V_0 - \epsilon\rangle &= \frac{1}{\sqrt{2mc^2(mc^2 - \epsilon)}} \frac{1}{\sqrt{2mc^2(mc^2 + \epsilon)}} \begin{bmatrix} -ic\hbar k & mc^2 - \epsilon \end{bmatrix} \begin{bmatrix} ic\hbar k \\ mc^2 + \epsilon \end{bmatrix} \\
&= \frac{2(c\hbar k)^2}{2mc^2\sqrt{(\hbar kc)^2}} \\
&= \operatorname{sgn}(k)\frac{\hbar k}{mc}.
\end{aligned}
\tag{1.24}
$$

Part 3.

$$
\begin{aligned}
\hat{v} &= \frac{1}{i\hbar}\left[\hat{x}, H\right] \\
&= \frac{1}{i\hbar}\left[\hat{x}, mc^2\sigma_z + V_0 + c\hat{p}\sigma_x\right] \\
&= \frac{c\sigma_x}{i\hbar}\left[\hat{x}, \hat{p}\right] \\
&= c\sigma_x.
\end{aligned}
\tag{1.25}
$$

Part 4. Acting against a completely general wavefunction the Hamiltonian action $H\psi$ is

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= mc^2\sigma_z\psi + V_0\psi + c\hat{p}\sigma_x\psi \\ &= mc^2\sigma_z\psi + V_0\psi - i\hbar c\sigma_x\frac{\partial\psi}{\partial x}. \end{aligned} \tag{1.26}$$

Conversely, the conjugate $(H\psi)^\dagger$ is

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t} = mc^2\psi^\dagger\sigma_z + V_0\psi^\dagger + i\hbar c\frac{\partial\psi^\dagger}{\partial x}\sigma_x. \tag{1.27}$$

These give

$$\begin{aligned} i\hbar\psi^\dagger\frac{\partial\psi}{\partial t} &= mc^2\psi^\dagger\sigma_z\psi + V_0\psi^\dagger\psi - i\hbar c\psi^\dagger\sigma_x\frac{\partial\psi}{\partial x} \\ -i\hbar\frac{\partial\psi^\dagger}{\partial t}\psi &= mc^2\psi^\dagger\sigma_z\psi + V_0\psi^\dagger\psi + i\hbar c\frac{\partial\psi^\dagger}{\partial x}\sigma_x\psi. \end{aligned} \tag{1.28}$$

Taking differences

$$\psi^\dagger\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\psi = -c\psi^\dagger\sigma_x\frac{\partial\psi}{\partial x} - c\frac{\partial\psi^\dagger}{\partial x}\sigma_x\psi, \tag{1.29}$$

or

$$0 = \frac{\partial}{\partial t}\left(\psi^\dagger\psi\right) + \frac{\partial}{\partial x}\left(c\psi^\dagger\sigma_x\psi\right). \tag{1.30}$$

The probability current still has the usual form $\rho = \psi^\dagger\psi = \psi_1^*\psi_1 + \psi_2^*\psi_2$, but the probability current with this representation of the Dirac Hamiltonian is

$$\begin{aligned} j &= c\psi^\dagger\sigma_x\psi \\ &= c\begin{bmatrix}\psi_1^* & \psi_2^*\end{bmatrix}\begin{bmatrix}\psi_2 \\ \psi_1\end{bmatrix} \\ &= c\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right). \end{aligned} \tag{1.31}$$