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Dirac delta function potential

Q:Dirac delta function potential Problem 2.24/2.25 [1] introduces a Dirac delta function potential

$$H = \frac{p^2}{2m} - V_0 \delta(x), \tag{1.1}$$

which vanishes after $t = 0$. Solve for the bound state for $t < 0$ and then the time evolution after that.

A: The first part of this problem was assigned back in phy356, where we solved this for a rectangular potential that had the limiting form of a delta function potential. However, this problem can be solved directly by considering the $|x| > 0$ and $x = 0$ regions separately.

For $|x| > 0$ Schrödinger's equation takes the form

$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}. \tag{1.2}$$

With

$$\kappa = \frac{\sqrt{-2mE}}{\hbar}, \tag{1.3}$$

this has solutions

$$\psi = e^{\pm \kappa x}. \tag{1.4}$$

For $x > 0$ we must have

$$\psi = a e^{-\kappa x}, \tag{1.5}$$

and for $x < 0$

$$\psi = b e^{\kappa x}. \tag{1.6}$$

requiring that ψ is continuous at $x = 0$ means $a = b$, or

$$\psi = \psi(0) e^{-\kappa |x|}. \tag{1.7}$$

For the $x = 0$ region, consider an interval $[-\epsilon, \epsilon]$ region around the origin. We must have

$$E \int_{-\epsilon}^{\epsilon} \psi(x) dx = \frac{-\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2} dx - V_0 \int_{-\epsilon}^{\epsilon} \delta(x)\psi(x) dx. \tag{1.8}$$

The RHS is zero

$$\begin{aligned} E \int_{-\epsilon}^{\epsilon} \psi(x) dx &= E \frac{e^{-\kappa(\epsilon)} - 1}{-\kappa} - E \frac{1 - e^{\kappa(-\epsilon)}}{\kappa} \\ &\to 0. \end{aligned} \tag{1.9}$$

That leaves

$$\begin{aligned} V_0 \int_{-\epsilon}^{\epsilon} \delta(x)\psi(x) dx &= \frac{-\hbar^2}{2m} \int_{-\epsilon}^{\epsilon} \frac{d^2\psi}{dx^2} dx \\ &= \frac{-\hbar^2}{2m} \left. \frac{d\psi}{dx} \right|_{-\epsilon}^{\epsilon} \\ &= \frac{-\hbar^2}{2m} \psi(0) \left(-\kappa e^{-\kappa(\epsilon)} - \kappa e^{\kappa(-\epsilon)} \right). \end{aligned} \tag{1.10}$$

In the $\epsilon \to 0$ limit this gives

$$V_0 = \frac{\hbar^2 \kappa}{m}. \tag{1.11}$$

Equating relations for κ we have

$$\kappa = \frac{m V_0}{\hbar^2} = \frac{\sqrt{-2mE}}{\hbar}, \tag{1.12}$$

or

$$E = -\frac{1}{2m} \left(\frac{m V_0}{\hbar} \right)^2, \tag{1.13}$$

with

$$\psi(x, t < 0) = C \exp\left(-iEt/\hbar - \kappa|x| \right). \tag{1.14}$$

The normalization requires

$$\begin{aligned} 1 &= 2|C|^2 \int_0^\infty e^{-2\kappa x} dx \\ &= 2|C|^2 \left. \frac{e^{-2\kappa x}}{-2\kappa} \right|_0^\infty \\ &= \frac{|C|^2}{\kappa}, \end{aligned} \tag{1.15}$$

so

$$\boxed{\psi(x, t < 0) = \frac{1}{\sqrt{\kappa}} \exp\left(-iEt/\hbar - \kappa|x| \right).} \tag{1.16}$$

There is only one bound state for such a potential. After turning off the potential, any plane wave

$$\psi(x,t) = e^{ikx - iE(k)t/\hbar}, \tag{1.17}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar}, \tag{1.18}$$

is a solution. In particular, at $t = 0$, the wave packet

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} A(k) dk, \tag{1.19}$$

is a solution. To solve for $A(k)$, we require

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} A(k) dk = \frac{1}{\sqrt{\kappa}} e^{-\kappa|x|}, \tag{1.20}$$

or

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi\kappa}} \int_{-\infty}^{\infty} e^{-ikx} e^{-mV_0|x|/\hbar^2} dx.} \tag{1.21}$$

The initial time state established by the delta function potential evolves as

$$\boxed{\psi(x,t>0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - i\hbar k^2 t/2m} A(k) dk.} \tag{1.22}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1