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Duality transformation

In a discussion of Dirac's monopoles, [1] introduces a duality transformation, forming electric and magnetic fields by forming a rotation that combines a different pair of electric and magnetic fields. In SI units that transformation becomes

$$\begin{bmatrix} \boldsymbol{\mathcal{E}} \\ \eta \boldsymbol{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{E}}' \\ \eta \boldsymbol{\mathcal{H}}' \end{bmatrix} \tag{1.1a}$$

$$\begin{bmatrix} \boldsymbol{\mathcal{D}} \\ \boldsymbol{\mathcal{B}}/\eta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{D}}' \\ \boldsymbol{\mathcal{B}}'/\eta \end{bmatrix}, \tag{1.1b}$$

where $\eta = \sqrt{\mu_0/\epsilon_0}$. It is left as an exercise to the reader to show that application of these to Maxwell's equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = \rho_\mathrm{e}/\epsilon_0 \tag{1.2a}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{H}} = \rho_\mathrm{m}/\mu_0 \tag{1.2b}$$
$$-\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} - \partial_t \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{J}}_\mathrm{m} \tag{1.2c}$$
$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}} - \partial_t \boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{J}}_\mathrm{e}, \tag{1.2d}$$

determine a similar relation between the sources. That transformation of Maxwell's equation is

$$\boldsymbol{\nabla} \cdot \left(\cos\theta \boldsymbol{\mathcal{E}}' + \sin\theta \eta \boldsymbol{\mathcal{H}}' \right) = \rho_\mathrm{e}/\epsilon_0 \tag{1.3a}$$
$$\boldsymbol{\nabla} \cdot \left(-\sin\theta \boldsymbol{\mathcal{E}}'/\eta + \cos\theta \boldsymbol{\mathcal{H}}' \right) = \rho_\mathrm{m}/\mu_0 \tag{1.3b}$$
$$-\boldsymbol{\nabla} \times \left(\cos\theta \boldsymbol{\mathcal{E}}' + \sin\theta \eta \boldsymbol{\mathcal{H}}' \right) - \partial_t \left(-\sin\theta \eta \boldsymbol{\mathcal{D}}' + \cos\theta \boldsymbol{\mathcal{B}}' \right) = \boldsymbol{\mathcal{J}}_\mathrm{m} \tag{1.3c}$$
$$\boldsymbol{\nabla} \times \left(-\sin\theta \boldsymbol{\mathcal{E}}'/\eta + \cos\theta \boldsymbol{\mathcal{H}}' \right) - \partial_t \left(\cos\theta \boldsymbol{\mathcal{D}}' + \sin\theta \boldsymbol{\mathcal{B}}'/\eta \right) = \boldsymbol{\mathcal{J}}_\mathrm{e}. \tag{1.3d}$$

A bit of rearranging gives

$$\begin{bmatrix} \eta \rho_\mathrm{e} \\ \rho_\mathrm{m} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \eta \rho_\mathrm{e}' \\ \rho_\mathrm{m}' \end{bmatrix} \tag{1.4a}$$

$$\begin{bmatrix} \eta \boldsymbol{\mathcal{J}}_\mathrm{e} \\ \boldsymbol{\mathcal{J}}_\mathrm{m} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \eta \boldsymbol{\mathcal{J}}_\mathrm{e}' \\ \boldsymbol{\mathcal{J}}_\mathrm{m}' \end{bmatrix}. \tag{1.4b}$$

For example, with $\rho_{\mathrm{m}} = \boldsymbol{\mathcal{J}}_{\mathrm{m}} = 0$, and $\theta = \pi/2$, the transformation of sources is

$$
\begin{aligned}
\rho_{\mathrm{e}}' &= 0 \\
\boldsymbol{\mathcal{J}}_{\mathrm{e}}' &= 0 \\
\rho_{\mathrm{m}}' &= \eta\rho_{\mathrm{e}} \\
\boldsymbol{\mathcal{J}}_{\mathrm{m}}' &= \eta\boldsymbol{\mathcal{J}}_{\mathrm{e}},
\end{aligned}
\tag{1.5}
$$

and Maxwell's equations then have only magnetic sources

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}}' = 0 \tag{1.6a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{H}}' = \rho_{\mathrm{m}}'/\mu_0 \tag{1.6b}$$

$$-\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}' - \partial_t \boldsymbol{\mathcal{B}}' = \boldsymbol{\mathcal{J}}_{\mathrm{m}}' \tag{1.6c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}}' - \partial_t \boldsymbol{\mathcal{D}}' = 0. \tag{1.6d}$$

Of this relation Jackson points out that "The invariance of the equations of electrodynamics under duality transformations shows that it is a matter of convention to speak of a particle possessing an electric charge, but not magnetic charge." This is an interesting comment, and worth some additional thought.

Bibliography

[1] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. 1