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Dynamics of non-Hermitian Hamiltonian

Exercise 1.1 Dynamics of non-Hermitian Hamiltonian ([1] *pr. 2.2*)

Revisiting an earlier Hamiltonian, but assuming it was entered incorrectly as

$$H = H_{11} \left|1\right\rangle \left\langle 1\right| + H_{22} \left|2\right\rangle \left\langle 2\right| + H_{12} \left|1\right\rangle \left\langle 2\right| . \tag{1.1}$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most generaqtl time-dependent problem using an illegal Hamiltonian of this kind. You may assume that $H_{11} = H_{22}$ for simplicity.

Answer for Exercise 1.1

In matrix form this Hamiltonian is

$$\begin{aligned} H &= \begin{bmatrix} \left\langle 1\right| H \left|1\right\rangle & \left\langle 1\right| H \left|2\right\rangle \\ \left\langle 2\right| H \left|1\right\rangle & \left\langle 2\right| H \left|2\right\rangle \end{bmatrix} \\ &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix} . \end{aligned} \tag{1.2}$$

This is not a Hermitian operator. What is the physical implication of this non-Hermicity? Consider the simpler case where $H_{11} = H_{22}$. Such a Hamiltonian has the form

$$H = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} . \tag{1.3}$$

This has only one unique eigenvector ($(1, 0)$, but we can still solve the time evolution equation

$$i \hbar \frac{\partial U}{\partial t} = H U, \tag{1.4}$$

since for constant H, we have

$$U = e^{-i H t/\hbar}. \tag{1.5}$$

To exponentiate, note that we have

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & n a^{n-1} b \\ 0 & a^n \end{bmatrix}. \tag{1.6}$$

To prove the induction, the $n = 2$ case follows easily

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}, \tag{1.7}$$

as does the general case

$$\begin{bmatrix} a^n & n a^{n-1} b \\ 0 & a^n \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^{n+1} & (n+1) a^n b \\ 0 & a^{n+1} \end{bmatrix}. \tag{1.8}$$

The exponential sum is thus

$$e^{H\tau} = \begin{bmatrix} e^{a\tau} & 0 + \frac{b\tau}{1!} + \frac{2ab\tau^2}{2!} + \frac{3a^2 b\tau^3}{3!} + \cdots \\ 0 & e^{a\tau} \end{bmatrix}. \tag{1.9}$$

That sum simplifies to

$$\begin{aligned} &\frac{b\tau}{0!} + \frac{ab\tau^2}{1!} + \frac{a^2 b\tau^3}{2!} + \cdots \\ &= b\tau \left(1 + \frac{a\tau}{1!} + \frac{(a\tau)^2}{2!} + \cdots \right) \\ &= b\tau e^{a\tau}. \end{aligned} \tag{1.10}$$

The exponential is thus

$$\begin{aligned} e^{H\tau} &= \begin{bmatrix} e^{a\tau} & b\tau e^{a\tau} \\ 0 & e^{a\tau} \end{bmatrix} \\ &= \begin{bmatrix} 1 & b\tau \\ 0 & 1 \end{bmatrix} e^{a\tau}. \end{aligned} \tag{1.11}$$

In particular

$$\begin{aligned} U &= e^{-iHt/\hbar} \\ &= \begin{bmatrix} 1 & -ibt/\hbar \\ 0 & 1 \end{bmatrix} e^{-iat/\hbar}. \end{aligned} \tag{1.12}$$

We can verify that this is a solution to eq. (1.4). The left hand side is

$$\begin{aligned} i\hbar \frac{\partial U}{\partial t} &= i\hbar \begin{bmatrix} -ia/\hbar & -ib/\hbar + (-ibt/\hbar)(-ia/\hbar) \\ 0 & -ia/\hbar \end{bmatrix} e^{-iat/\hbar} \\ &= \begin{bmatrix} a & b - iabt/\hbar \\ 0 & a \end{bmatrix} e^{-iat/\hbar}, \end{aligned} \tag{1.13}$$

and for the right hand side

$$\begin{aligned} HU &= \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & -ibt/\hbar \\ 0 & 1 \end{bmatrix} e^{-iat/\hbar} \\ &= \begin{bmatrix} a & b - iabt/\hbar \\ 0 & a \end{bmatrix} e^{-iat/\hbar} \\ &= i\hbar \frac{\partial U}{\partial t}. \qquad \square \end{aligned} \tag{1.14}$$

While the Schrödinger is satisfied, we don't have the unitary invertion physical property that is desired for the time evolution operator U. Namely

$$\begin{aligned} U^\dagger U &= \begin{bmatrix} 1 & 0 \\ ibt/\hbar & 1 \end{bmatrix} e^{iat/\hbar} \begin{bmatrix} 1 & -ibt/\hbar \\ 0 & 1 \end{bmatrix} e^{-iat/\hbar} \\ &= \begin{bmatrix} 1 & -ibt/\hbar \\ ibt/\hbar & (bt)^2/\hbar^2 \end{bmatrix} \\ &\neq I. \end{aligned} \tag{1.15}$$

We required $U^\dagger U = I$ for the time evolution operator, but don't have that property for this non-Hermitian Hamiltonian.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1