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Plane wave ground state expectation

Problem [1] 2.18 is, for a 1D SHO, show that

$$\left\langle 0 \right\rvert e^{ikx} \left\lvert 0 \right\rangle = \exp\left(-k^2 \left\langle 0 \right\rvert x^2 \left\lvert 0 \right\rangle /2 \right). \tag{1.1}$$

Despite the simple appearance of this problem, I found this quite involved to show. To do so, start with a series expansion of the expectation

$$\left\langle 0 \right\rvert e^{ikx} \left\lvert 0 \right\rangle = \sum_{m=0}^\infty \frac{(ik)^m}{m!} \left\langle 0 \right\rvert x^m \left\lvert 0 \right\rangle. \tag{1.2}$$

Let

$$X = \left(a + a^\dagger \right), \tag{1.3}$$

so that

$$x = \sqrt{\frac{\hbar}{2 \omega m}} X = \frac{x_0}{\sqrt{2}} X. \tag{1.4}$$

Consider the first few values of $\left\langle 0 \right\rvert X^n \left\lvert 0 \right\rangle$

$$\begin{aligned}
\left\langle 0 \right\rvert X \left\lvert 0 \right\rangle &= \left\langle 0 \right\rvert \left(a + a^\dagger \right) \left\lvert 0 \right\rangle \\
&= \left\langle 0 \vert 1 \right\rangle \\
&= 0,
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
\left\langle 0 \right\rvert X^2 \left\lvert 0 \right\rangle &= \left\langle 0 \right\rvert \left(a + a^\dagger \right)^2 \left\lvert 0 \right\rangle \\
&= \left\langle 1 \vert 1 \right\rangle \\
&= 1,
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
\left\langle 0 \right\rvert X^3 \left\lvert 0 \right\rangle &= \left\langle 0 \right\rvert \left(a + a^\dagger \right)^3 \left\lvert 0 \right\rangle \\
&= \left\langle 1 \right\rvert \left(\sqrt{2} \left\lvert 2 \right\rangle + \left\lvert 0 \right\rangle \right) \\
&= 0.
\end{aligned} \tag{1.7}$$

Whenever the power n in X^n is even, the braket can be split into a bra that has only contributions from odd eigenstates and a ket with even eigenstates. We conclude that $\langle 0| X^n |0\rangle = 0$ when n is odd.

Noting that $\langle 0| x^2 |0\rangle = x_0^2/2$, this leaves

$$\begin{aligned}\langle 0| e^{ikx} |0\rangle &= \sum_{m=0}^{\infty} \frac{(ik)^{2m}}{(2m)!} \langle 0| x^{2m} |0\rangle \\ &= \sum_{m=0}^{\infty} \frac{(ik)^{2m}}{(2m)!} \left(\frac{x_0^2}{2}\right)^m \langle 0| X^{2m} |0\rangle \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(-k^2 \langle 0| x^2 |0\rangle\right)^m \langle 0| X^{2m} |0\rangle .\end{aligned} \tag{1.8}$$

This problem is now reduced to showing that

$$\frac{1}{(2m)!} \langle 0| X^{2m} |0\rangle = \frac{1}{m!\, 2^m}, \tag{1.9}$$

or

$$\begin{aligned}\langle 0| X^{2m} |0\rangle &= \frac{(2m)!}{m!\, 2^m} \\ &= \frac{(2m)(2m-1)(2m-2)\cdots(2)(1)}{2^m m!} \\ &= \frac{2^m (m)(2m-1)(m-1)(2m-3)(m-2)\cdots(2)(3)(1)(1)}{2^m m!} \\ &= (2m-1)!!,\end{aligned} \tag{1.10}$$

where $n!! = n(n-2)(n-4)\cdots$.

It looks like $\langle 0| X^{2m} |0\rangle$ can be expanded by inserting an identity operator and proceeding recursively, like

$$\begin{aligned}\langle 0| X^{2m} |0\rangle &= \langle 0| X^2 \left(\sum_{n=0}^{\infty} |n\rangle \langle n|\right) X^{2m-2} |0\rangle \\ &= \langle 0| X^2 (|0\rangle \langle 0| + |2\rangle \langle 2|) X^{2m-2} |0\rangle \\ &= \langle 0| X^{2m-2} |0\rangle + \langle 0| X^2 |2\rangle \langle 2| X^{2m-2} |0\rangle .\end{aligned} \tag{1.11}$$

This has made use of the observation that $\langle 0| X^2 |n\rangle = 0$ for all $n \ne 0, 2$. The remaining term includes the factor

$$\begin{aligned}\langle 0| X^2 |2\rangle &= \langle 0| \left(a + a^\dagger\right)^2 |2\rangle \\ &= \left(\langle 0| + \sqrt{2} \langle 2|\right) |2\rangle \\ &= \sqrt{2},\end{aligned} \tag{1.12}$$

Since $\sqrt{2}\,|2\rangle = \left(a^\dagger\right)^2 |0\rangle$, the expectation of interest can be written

$$\langle 0|\, X^{2m}\, |0\rangle = \langle 0|\, X^{2m-2}\, |0\rangle + \langle 0|\, a^2 X^{2m-2}\, |0\rangle \,. \tag{1.13}$$

How do we expand the second term. Let's look at how a and X commute

$$\begin{aligned} aX &= [a, X] + Xa \\ &= \left[a, a + a^\dagger\right] + Xa \\ &= \left[a, a^\dagger\right] + Xa \\ &= 1 + Xa, \end{aligned} \tag{1.14}$$

$$\begin{aligned} a^2 X &= a\,(aX) \\ &= a\,(1 + Xa) \\ &= a + aXa \\ &= a + (1 + Xa)\,a \\ &= 2a + Xa^2. \end{aligned} \tag{1.15}$$

Proceeding to expand $a^2 X^n$ we find

$$\begin{aligned} a^2 X^3 &= 6X + 6X^2 a + X^3 a^2 \\ a^2 X^4 &= 12X^2 + 8X^3 a + X^4 a^2 \\ a^2 X^5 &= 20X^3 + 10X^4 a + X^5 a^2 \\ a^2 X^6 &= 30X^4 + 12X^5 a + X^6 a^2. \end{aligned} \tag{1.16}$$

It appears that we have

$$\left[a^2 X^n, X^n a^2\right] = \beta_n X^{n-2} + 2nX^{n-1} a, \tag{1.17}$$

where

$$\beta_n = \beta_{n-1} + 2(n-1), \tag{1.18}$$

and $\beta_2 = 2$. Some goofing around shows that $\beta_n = n(n-1)$, so the induction hypothesis is

$$\left[a^2 X^n, X^n a^2\right] = n(n-1)X^{n-2} + 2nX^{n-1} a. \tag{1.19}$$

Let's check the induction

$$\begin{aligned} a^2 X^{n+1} &= a^2 X^n X \\ &= \left(n(n-1)X^{n-2} + 2nX^{n-1} a + X^n a^2\right) X \\ &= n(n-1)X^{n-1} + 2nX^{n-1} aX + X^n a^2 X \\ &= n(n-1)X^{n-1} + 2nX^{n-1}\,(1 + Xa) + X^n\,\left(2a + Xa^2\right) \\ &= n(n-1)X^{n-1} + 2nX^{n-1} + 2nX^n a + 2X^n a + X^{n+1} a^2 \\ &= X^{n+1} a^2 + (2 + 2n)X^n a + (2n + n(n-1))\,X^{n-1} \\ &= X^{n+1} a^2 + 2(n+1)X^n a + (n+1)nX^{n-1}, \end{aligned} \tag{1.20}$$

which concludes the induction, giving

$$\langle 0| a^2 X^n |0\rangle = n(n-1) \langle 0| X^{n-2} |0\rangle \,, \tag{1.21}$$

and

$$\langle 0| X^{2m} |0\rangle = \langle 0| X^{2m-2} |0\rangle + (2m-2)(2m-3) \langle 0| X^{2m-4} |0\rangle \,. \tag{1.22}$$

Let

$$\sigma_n = \langle 0| X^n |0\rangle \,, \tag{1.23}$$

so that the recurrence relation, for $2n \geq 4$ is

$$\sigma_{2n} = \sigma_{2n-2} + (2n-2)(2n-3)\sigma_{2n-4} \tag{1.24}$$

We want to show that this simplifies to

$$\sigma_{2n} = (2n-1)!! \tag{1.25}$$

The first values are

$$\sigma_0 = \langle 0| X^0 |0\rangle = 1 \tag{1.26a}$$

$$\sigma_2 = \langle 0| X^2 |0\rangle = 1 \tag{1.26b}$$

which gives us the right result for the first term in the induction

$$\begin{aligned} \sigma_4 &= \sigma_2 + 2 \times 1 \times \sigma_0 \\ &= 1 + 2 \\ &= 3!! \end{aligned} \tag{1.27}$$

For the general induction term, consider

$$\begin{aligned} \sigma_{2n+2} &= \sigma_{2n} + 2n(2n-1)\sigma_{2n-2} \\ &= (2n-1)!! + 2n(2n-1)(2n-3)!! \\ &= (2n+1)(2n-1)!! \\ &= (2n+1)!! \,, \end{aligned} \tag{1.28}$$

which completes the final induction. That was also the last thing required to complete the proof, so we are done!

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1