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Relation of probability flux to momentum

In [1] it is mentioned that the probability flux

$$\mathbf{j}(\mathbf{x}, t) = -\frac{i\hbar}{2m} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right), \tag{1.1}$$

is related to the momentum expectation at a given time by the integral of the flux over all space

$$\int d^3 x \mathbf{j}(\mathbf{x}, t) = \frac{\langle \mathbf{p} \rangle_t}{m}. \tag{1.2}$$

That wasn't obvious to me at a glance, however, this can be seen by recasting the integral in bra-ket form. Let

$$\psi(\mathbf{x}, t) = \left\langle \mathbf{x} \vert \psi(t) \right\rangle, \tag{1.3}$$

and note that the momentum portions of the flux can be written as

$$-i\hbar \boldsymbol{\nabla} \psi(\mathbf{x}, t) = \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle. \tag{1.4}$$

The current is therefore

$$\begin{aligned}
\mathbf{j}(\mathbf{x}, t) &= \frac{1}{2m} \left(\psi^* \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle + \psi \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle^* \right) \\
&= \frac{1}{2m} \left(\left\langle \mathbf{x} \vert \psi(t) \right\rangle^* \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle + \left\langle \mathbf{x} \vert \psi(t) \right\rangle \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle^* \right) \\
&= \frac{1}{2m} \left(\left\langle \psi(t) \vert \mathbf{x} \right\rangle \left\langle \mathbf{x} \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle + \left\langle \psi(t) \right\rvert \mathbf{p} \left\lvert \mathbf{x} \right\rangle \left\langle \mathbf{x} \vert \psi(t) \right\rangle \right).
\end{aligned} \tag{1.5}$$

Integrating and noting that the spatial identity is $1 = \int d^3 x \left\lvert \mathbf{x} \right\rangle \left\langle \mathbf{x} \right\rvert$, we have

$$\int d^3 x \mathbf{j}(\mathbf{x}, t) = \left\langle \psi(t) \right\rvert \mathbf{p} \left\lvert \psi(t) \right\rangle, \tag{1.6}$$

This is just the expectation of $\mathbf{p}$ with respect to a specific time-instance state.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1