Peeter Joot
peeter.joot@gmail.com

Gauge transformation of free particle Hamiltonian

Exercise 1.1

Given a gauge transformation of the free particle Hamiltonian to

$$H = \frac{1}{2m}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi} + e\phi, \tag{1.1}$$

where

$$\boldsymbol{\Pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}, \tag{1.2}$$

calculate $m d\mathbf{x}/dt$, $\left[\Pi_i, \Pi_j\right]$, and $m d^2\mathbf{x}/dt^2$, where $\mathbf{x}$ is the Heisenberg picture position operator, and the fields are functions only of position $\phi = \phi(\mathbf{x})$, $\mathbf{A} = \mathbf{A}(\mathbf{x})$.

Answer for Exercise 1.1

The final results for these calculations are found in [1], but seem worth deriving to exercise our commutator muscles.

Heisenberg picture velocity operator The first order of business is the Heisenberg picture velocity operator, but first note

$$\begin{aligned}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi} &= \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \\ &= \mathbf{p}^2 - \frac{e}{c}\left(\mathbf{A} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{A}\right) + \frac{e^2}{c^2}\mathbf{A}^2.\end{aligned} \tag{1.3}$$

The time evolution of the Heisenberg picture position operator is therefore

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \frac{1}{i\hbar}\left[\mathbf{x}, H\right] \\ &= \frac{1}{i\hbar 2m}\left[\mathbf{x}, \boldsymbol{\Pi}^2\right] \\ &= \frac{1}{i\hbar 2m}\left[\mathbf{x}, \mathbf{p}^2 - \frac{e}{c}\left(\mathbf{A} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{A}\right) + \frac{e^2}{c^2}\mathbf{A}^2\right] \\ &= \frac{1}{i\hbar 2m}\left(\left[\mathbf{x}, \mathbf{p}^2\right] - \frac{e}{c}\left[\mathbf{x}, \mathbf{A} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{A}\right]\right).\end{aligned} \tag{1.4}$$

For the $\mathbf{p}^2$ commutator we have

$$\begin{aligned} \left[x_r, \mathbf{p}^2\right] &= i\hbar \frac{\partial \mathbf{p}^2}{\partial p_r} \\ &= 2i\hbar p_r, \end{aligned} \tag{1.5}$$

or

$$\left[\mathbf{x}, \mathbf{p}^2\right] = 2i\hbar \mathbf{p}. \tag{1.6}$$

Computing the remaining commutator, we've got

$$\begin{aligned} \left[x_r, \mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}\right] &= x_r p_s A_s - p_s A_s x_r \\ &\quad + x_r A_s p_s - A_s p_s x_r \\ &= \left(\left[x_r, p_s\right] + p_s x_r\right) A_s - p_s A_s x_r \\ &\quad + x_r A_s p_s - A_s \left(\left[p_s, x_r\right] + x_r p_s\right) \\ &= \left[x_r, p_s\right] A_s + \cancel{p_s A_s x_r - p_s A_s x_r} \\ &\quad + \cancel{x_r A_s p_s - x_r A_s p_s} + A_s \left[x_r, p_s\right] \\ &= 2i\hbar \delta_{rs} A_s \\ &= 2i\hbar A_r, \end{aligned} \tag{1.7}$$

so

$$\left[\mathbf{x}, \mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}\right] = 2i\hbar \mathbf{A}. \tag{1.8}$$

Assembling these results gives

$$\boxed{\frac{d\mathbf{x}}{dt} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) = \frac{1}{m}\boldsymbol{\Pi},} \tag{1.9}$$

as asserted in the text.

Kinetic Momentum commutators

$$\begin{aligned} \left[\Pi_r, \Pi_s\right] &= \left[p_r - eA_r/c, p_s - eA_s/c\right] \\ &= \cancel{\left[p_r, p_s\right]} - \frac{e}{c}\left(\left[p_r, A_s\right] + \left[A_r, p_s\right]\right) + \frac{e^2}{c^2}\cancel{\left[A_r, A_s\right]}. \\ &= -\frac{e}{c}\left((-i\hbar)\frac{\partial A_s}{\partial x_r} + (i\hbar)\frac{\partial A_r}{\partial x_s}\right) \\ &= -\frac{ie\hbar}{c}\left(-\frac{\partial A_s}{\partial x_r} + \frac{\partial A_r}{\partial x_s}\right). \\ &= -\frac{ie\hbar}{c}\epsilon_{tsr} B_t, \end{aligned} \tag{1.10}$$

or

$$\boxed{\left[\Pi_r, \Pi_s\right] = \frac{ie\hbar}{c}\epsilon_{rst} B_t.} \tag{1.11}$$

Quantum Lorentz force For the force equation we have

$$\begin{aligned} m\frac{d^2\mathbf{x}}{dt^2} &= \frac{d\boldsymbol{\Pi}}{dt} \\ &= \frac{1}{i\hbar}[\boldsymbol{\Pi}, H] \\ &= \frac{1}{i\hbar 2m}\left[\boldsymbol{\Pi}, \boldsymbol{\Pi}^2\right] + \frac{1}{i\hbar}\left[\boldsymbol{\Pi}, e\phi\right]. \end{aligned} \tag{1.12}$$

For the ϕ commutator consider one component

$$\begin{aligned} \left[\Pi_r, e\phi\right] &= e\left[p_r - \frac{e}{c}A_r, \phi\right] \\ &= e\left[p_r, \phi\right] \\ &= e(-i\hbar)\frac{\partial\phi}{\partial x_r}, \end{aligned} \tag{1.13}$$

or

$$\frac{1}{i\hbar}[\boldsymbol{\Pi}, e\phi] = -e\boldsymbol{\nabla}\phi = e\mathbf{E}. \tag{1.14}$$

For the $\boldsymbol{\Pi}^2$ commutator I initially did this the hard way (it took four notebook pages, plus two for a false start.) Realizing that I didn't use eq. (1.11) for that expansion was the clue to doing this more expediently.

Considering a single component

$$\begin{aligned} \left[\Pi_r, \boldsymbol{\Pi}^2\right] &= [\Pi_r, \Pi_s\Pi_s] \\ &= \Pi_r\Pi_s\Pi_s - \Pi_s\Pi_s\Pi_r \\ &= \left([\Pi_r, \Pi_s] + \cancel{\Pi_s\Pi_r}\right)\Pi_s - \Pi_s\left([\Pi_s, \Pi_r] + \cancel{\Pi_r\Pi_s}\right) \\ &= i\hbar\frac{e}{c}\epsilon_{rst}\left(B_t\Pi_s + \Pi_s B_t\right), \end{aligned} \tag{1.15}$$

or

$$\begin{aligned} \frac{1}{i\hbar 2m}\left[\boldsymbol{\Pi}, \boldsymbol{\Pi}^2\right] &= \frac{e}{2mc}\epsilon_{rst}\mathbf{e}_r\left(B_t\Pi_s + \Pi_s B_t\right) \\ &= \frac{e}{2mc}\left(\boldsymbol{\Pi}\times\mathbf{B} - \mathbf{B}\times\boldsymbol{\Pi}\right). \end{aligned} \tag{1.16}$$

Putting all the pieces together we've got the quantum equivalent of the Lorentz force equation

$$\boxed{m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E} + \frac{e}{2c}\left(\frac{d\mathbf{x}}{dt}\times\mathbf{B} - \mathbf{B}\times\frac{d\mathbf{x}}{dt}\right).} \tag{1.17}$$

While this looks equivalent to the classical result, all the vectors here are Heisenberg picture operators dependent on position.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1