Peeter Joot
peeter.joot@gmail.com

Gauge transformed probability current

Exercise 1.1 Gauge transformed probability current ([1] *pr. 2.37 (b)*)

For the gauge transformed Schrödinger equation

$$\frac{1}{2m}\boldsymbol{\Pi}(\mathbf{x}) \cdot \boldsymbol{\Pi}(\mathbf{x})\psi(\mathbf{x}, t) + e\phi(\mathbf{x})\psi(\mathbf{x}, t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t), \tag{1.1}$$

where

$$\boldsymbol{\Pi}(\mathbf{x}) = -i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}(\mathbf{x}), \tag{1.2}$$

find the probability current defined by

$$\frac{\partial \psi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}. \tag{1.3}$$

Answer for Exercise 1.1

Equation eq. (1.1) and its conjugate are

$$\begin{aligned}\frac{1}{2m}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi}\psi + e\phi\psi &= i\hbar\frac{\partial \psi}{\partial t} \\ \frac{1}{2m}\boldsymbol{\Pi}^* \cdot \boldsymbol{\Pi}^*\psi^* + e\phi\psi^* &= -i\hbar\frac{\partial \psi^*}{\partial t}\end{aligned} \tag{1.4}$$

which can be used immediately in a chain rule expansion of the probability time derivative

$$\begin{aligned}i\hbar\frac{\partial \rho}{\partial t} &= i\hbar\psi^*\frac{\partial \psi}{\partial t} + i\hbar\psi\frac{\partial \psi^*}{\partial t} \\ &= \psi^*\left(\frac{1}{2m}\boldsymbol{\Pi} \cdot \boldsymbol{\Pi}\psi + e\phi\psi\right) - \psi\left(\frac{1}{2m}\boldsymbol{\Pi}^* \cdot \boldsymbol{\Pi}^*\psi^* + e\phi\psi^*\right) \\ &= \frac{1}{2m}\left(\psi^*\boldsymbol{\Pi} \cdot \boldsymbol{\Pi}\psi - \psi\boldsymbol{\Pi}^* \cdot \boldsymbol{\Pi}^*\psi^*\right).\end{aligned} \tag{1.5}$$

We have a difference of conjugates, so can get away with expanding just the first term

$$
\begin{aligned}
\psi^* \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi &= \psi^* \psi \\
&= \psi^* \left(-i\hbar \boldsymbol{\nabla} - \frac{e}{c} \mathbf{A} \right) \cdot \left(-i\hbar \boldsymbol{\nabla} - \frac{e}{c} \mathbf{A} \right) \psi \\
&= \psi^* \left(-\hbar^2 \boldsymbol{\nabla}^2 + \frac{i\hbar e}{c} \left(\mathbf{A} \cdot \boldsymbol{\nabla} + \boldsymbol{\nabla} \cdot \mathbf{A} \right) + \frac{e^2}{c^2} \mathbf{A}^2 \right) \psi.
\end{aligned} \tag{1.6}
$$

Note that in the directional derivative terms, the gradient operates on everything to its right, including $\mathbf{A}$. Also note that the last term has no imaginary component, so it will not contribute to the difference of conjugates.

This gives

$$
\begin{aligned}
\psi^* \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi - \psi \boldsymbol{\Pi}^* \cdot \boldsymbol{\Pi}^* \psi^* &= \psi^* \left(-\hbar^2 \boldsymbol{\nabla}^2 \psi + \frac{i\hbar e}{c} \left(\mathbf{A} \cdot \boldsymbol{\nabla} \psi + \boldsymbol{\nabla} \cdot (\mathbf{A} \psi) \right) \right) \\
&\quad - \psi \left(-\hbar^2 \boldsymbol{\nabla}^2 \psi^* - \frac{i\hbar e}{c} \left(\mathbf{A} \cdot \boldsymbol{\nabla} \psi^* + \boldsymbol{\nabla} \cdot (\mathbf{A} \psi^*) \right) \right) \\
&= -\hbar^2 \left(\psi^* \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \psi^* \right) \\
&\quad + \frac{i\hbar e}{c} \left(\psi^* \mathbf{A} \cdot \boldsymbol{\nabla} \psi + \psi^* \boldsymbol{\nabla} \cdot (\mathbf{A} \psi) + \psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^* + \psi \boldsymbol{\nabla} \cdot (\mathbf{A} \psi^*) \right)
\end{aligned} \tag{1.7}
$$

The first term is recognized as a divergence

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) &= \psi^* \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi + \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \psi^* - \psi \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi^* - \boldsymbol{\nabla} \psi^* \cdot \boldsymbol{\nabla} \psi \\
&= \psi^* \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \psi^*.
\end{aligned} \tag{1.8}
$$

The second term can also be factored into a divergence operation

$$
\begin{aligned}
&\psi^* \mathbf{A} \cdot \boldsymbol{\nabla} \psi + \psi^* \boldsymbol{\nabla} \cdot (\mathbf{A} \psi) + \psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^* + \psi \boldsymbol{\nabla} \cdot (\mathbf{A} \psi^*) \\
&\quad = \left(\psi^* \mathbf{A} \cdot \boldsymbol{\nabla} \psi + \psi \boldsymbol{\nabla} \cdot (\mathbf{A} \psi^*) \right) + \left(\psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^* + \psi^* \boldsymbol{\nabla} \cdot (\mathbf{A} \psi) \right) \\
&\quad = 2 \boldsymbol{\nabla} \cdot \left(\mathbf{A} \psi \psi^* \right)
\end{aligned} \tag{1.9}
$$

Putting all the pieces back together we have

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{1}{2mi\hbar} \left(\psi^* \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi - \psi \boldsymbol{\Pi}^* \cdot \boldsymbol{\Pi}^* \psi^* \right) \\
&= \boldsymbol{\nabla} \cdot \frac{1}{2mi\hbar} \left(-\hbar^2 \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) + \frac{i\hbar e}{c} 2 \mathbf{A} \psi \psi^* \right) \\
&= \boldsymbol{\nabla} \cdot \left(\frac{i\hbar}{2m} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) + \frac{e}{mc} \mathbf{A} \psi \psi^* \right).
\end{aligned} \tag{1.10}
$$

From eq. (1.3), the probability current must be

$$
\mathbf{j} = \frac{\hbar}{2im} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) - \frac{e}{mc} \mathbf{A} \psi \psi^*, \tag{1.11}
$$

or

$$
\boxed{\mathbf{j} = \frac{\hbar}{m} \operatorname{Im} \left(\psi^* \boldsymbol{\nabla} \psi \right) - \frac{e}{mc} \mathbf{A} \psi \psi^*.} \tag{1.12}
$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1