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Harmonic oscillator with energy shift

Exercise 1.1 Harmonic oscillator with energy shift. ([1] pr. 5.1)

Given a perturbed 1D SHO Hamiltonian

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 + \lambda b x, \tag{1.1}$$

calculate the first non-zero perturbation to the ground state energy. Then solve for that energy directly and compare.

Answer for Exercise 1.1

The first order energy shift is seen to be zero

$$\begin{aligned}\Delta_0^{(0)} &= V_{00} \\ &= \left\langle 0 \right\rvert b x \left\lvert 0 \right\rangle \\ &= \frac{x_0}{\sqrt{2}} \left\langle 0 \right\rvert a + a^\dagger \left\lvert 0 \right\rangle \\ &= \frac{x_0}{\sqrt{2}} \left\langle 0 | 1 \right\rangle \\ &= 0.\end{aligned} \tag{1.2}$$

The first order perturbation to the ground state is

$$\begin{aligned}\left\lvert 0^{(1)} \right\rangle &= \sum_{m \ne 0} \frac{\left\lvert m \right\rangle \left\langle m \right\rvert b x \left\lvert 0 \right\rangle}{\hbar\omega/2 - \hbar\omega(m - 1/2)} \\ &= -b \frac{x_0}{\sqrt{2}\hbar\omega} \sum_{m \ne 0} \frac{\left\lvert m \right\rangle \left\langle m | 1 \right\rangle}{m} \\ &= -b \frac{x_0}{\sqrt{2}\hbar\omega} \left\lvert 1 \right\rangle.\end{aligned} \tag{1.3}$$

The second order ground state energy perturbation is

$$\begin{aligned}
\Delta_0^{(2)} &= \langle 0| bx \left| 0^{(1)} \right\rangle \\
&= \frac{bx_0}{\sqrt{2}} \langle 0| a + a^\dagger \left(-b \frac{x_0}{\sqrt{2}\hbar\omega} |1\rangle \right) \\
&= \frac{bx_0}{\sqrt{2}} \left(-b \frac{x_0}{\sqrt{2}\hbar\omega} \right) \\
&= -\frac{b^2 x_0^2}{2\hbar\omega} \\
&= -\frac{b^2}{2\hbar\omega} \frac{\hbar}{m\omega} \\
&= -\frac{b^2}{2m\omega^2},
\end{aligned} \tag{1.4}$$

so the total energy perturbation up to second order is

$$\Delta_0 = -\lambda^2 \frac{b^2}{2m\omega^2}. \tag{1.5}$$

To compare to the exact result, rewrite the Hamiltonian as

$$\begin{aligned}
H &= \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 \left(x^2 + \frac{2\lambda bx}{m\omega^2} \right) \\
&= \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 \left(x + \frac{\lambda b}{m\omega^2} \right)^2 - \frac{1}{2} m\omega^2 \left(\frac{\lambda b}{m\omega^2} \right)^2.
\end{aligned} \tag{1.6}$$

The Hamiltonian is subject to a constant energy shift

$$\begin{aligned}
\Delta E &= -\frac{1}{2} m\omega^2 \frac{\lambda^2 b^2}{m^2\omega^4} \\
&= -\frac{\lambda^2 b^2}{2m\omega^2}.
\end{aligned} \tag{1.7}$$

This is an exact match with the second order perturbation result of eq. (1.5).

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1