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Heisenberg picture spin precession

Exercise 1.1 Heisenberg picture spin precession ([1] pr. 2.1)

For the spin Hamiltonian

$$\begin{aligned} H &= -\frac{eB}{mc} S_z \\ &= \omega S_z, \end{aligned} \tag{1.1}$$

express and solve the Heisenberg equations of motion for $S_x(t)$, $S_y(t)$, and $S_z(t)$.

Answer for Exercise 1.1

The equations of motion are of the form

$$\begin{aligned} \frac{dS_i^{\mathrm{H}}}{dt} &= \frac{1}{i\hbar} \left[S_i^{\mathrm{H}}, H \right] \\ &= \frac{1}{i\hbar} \left[U^\dagger S_i U, H \right] \\ &= \frac{1}{i\hbar} \left(U^\dagger S_i U H - H U^\dagger S_i U \right) \\ &= \frac{1}{i\hbar} U^\dagger \left(S_i H - H S_i \right) U \\ &= \frac{\omega}{i\hbar} U^\dagger \left[S_i, S_z \right] U. \end{aligned} \tag{1.2}$$

These are

$$\begin{aligned} \frac{dS_x^{\mathrm{H}}}{dt} &= -\omega U^\dagger S_y U \\ \frac{dS_y^{\mathrm{H}}}{dt} &= \omega U^\dagger S_x U \\ \frac{dS_z^{\mathrm{H}}}{dt} &= 0. \end{aligned} \tag{1.3}$$

To completely specify these equations, we need to expand $U(t)$, which is

$$
\begin{aligned}
U(t) &= e^{-iHt/\hbar} \\
&= e^{-i\omega S_z t/\hbar} \\
&= e^{-i\omega\sigma_z t/2} \\
&= \cos\left(\omega t/2\right) - i\sigma_z \sin\left(\omega t/2\right) \\
&= \begin{bmatrix} \cos\left(\omega t/2\right) - i\sin\left(\omega t/2\right) & 0 \\ 0 & \cos\left(\omega t/2\right) + i\sin\left(\omega t/2\right) \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix}.
\end{aligned}
\tag{1.4}
$$

The equations of motion can now be written out in full. To do so seems a bit silly since we also know that $S_x^{\mathrm{H}} = U^\dagger S_x U, S_y^{\mathrm{H}} U^\dagger S_x U$. However, if that is temporarily forgotten, we can show that the Heisenberg equations of motion can be solved for these too.

$$
\begin{aligned}
U^\dagger S_x U &= \frac{\hbar}{2} \begin{bmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix} \\
&= \frac{\hbar}{2} \begin{bmatrix} 0 & e^{i\omega t/2} \\ e^{-i\omega t/2} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix} \\
&= \frac{\hbar}{2} \begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix},
\end{aligned}
\tag{1.5}
$$

and

$$
\begin{aligned}
U^\dagger S_y U &= \frac{\hbar}{2} \begin{bmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix} \\
&= \frac{i\hbar}{2} \begin{bmatrix} 0 & -e^{i\omega t/2} \\ e^{-i\omega t/2} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix} \\
&= \frac{i\hbar}{2} \begin{bmatrix} 0 & -e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix}.
\end{aligned}
\tag{1.6}
$$

The equations of motion are now fully specified

$$
\begin{aligned}
\frac{dS_x^{\mathrm{H}}}{dt} &= -\frac{i\hbar\omega}{2} \begin{bmatrix} 0 & -e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \\
\frac{dS_y^{\mathrm{H}}}{dt} &= \frac{\hbar\omega}{2} \begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \\
\frac{dS_z^{\mathrm{H}}}{dt} &= 0.
\end{aligned}
\tag{1.7}
$$

Integration gives

$$
\begin{aligned}
S_x^{\mathrm{H}} &= \frac{\hbar}{2}\begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} + C \\
S_y^{\mathrm{H}} &= \frac{\hbar}{2}\begin{bmatrix} 0 & -ie^{i\omega t} \\ ie^{-i\omega t} & 0 \end{bmatrix} + C \\
S_z^{\mathrm{H}} &= C.
\end{aligned}
\tag{1.8}
$$

The integration constants are fixed by the boundary condition $S_i^{\mathrm{H}}(0) = S_i$, so

$$
\begin{aligned}
S_x^{\mathrm{H}} &= \frac{\hbar}{2}\begin{bmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \\
S_y^{\mathrm{H}} &= \frac{i\hbar}{2}\begin{bmatrix} 0 & -e^{i\omega t} \\ e^{-i\omega t} & 0 \end{bmatrix} \\
S_z^{\mathrm{H}} &= S_z.
\end{aligned}
\tag{1.9}
$$

Observe that these integrated values $S_x^{\mathrm{H}}, S_y^{\mathrm{H}}$ match eq. (1.5), and eq. (1.6) as expected.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1