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Lagrangian for magnetic portion of Lorentz force

In [1] it is claimed in an Aharonov-Bohm discussion that a Lagrangian modification to include electromagnetism is

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{e}{c}\mathbf{v} \cdot \mathbf{A}. \tag{1.1}$$

That can't be the full Lagrangian since there is no ϕ term, so what exactly do we get?

If you have somehow, like I did, forgot the exact form of the Euler-Lagrange equations (i.e. where do the dots go), then the derivation of those equations can come to your rescue. The starting point is the action

$$S = \int \mathcal{L}(x, \dot{x}, t) dt, \tag{1.2}$$

where the end points of the integral are fixed, and we assume we have no variation at the end points. The variational calculation is

$$\begin{aligned}
\delta S &= \int \delta \mathcal{L}(x, \dot{x}, t) dt \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} \right) dt \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \frac{dx}{dt} \right) dt \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \right) \delta x dt + \delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}.
\end{aligned} \tag{1.3}$$

The boundary term is killed after evaluation at the end points where the variation is zero. For the result to hold for all variations δx, we must have

$$\boxed{\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right).} \tag{1.4}$$

Now lets apply this to the Lagrangian at hand. For the position derivative we have

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{e}{c} v_j \frac{\partial A_j}{\partial x_i}. \tag{1.5}$$

For the canonical momentum term, assuming $\mathbf{A} = \mathbf{A}(\mathbf{x})$ we have

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} &= \frac{d}{dt}\left(m\dot{x}_i + \frac{e}{c}A_i \right) \\
&= m\ddot{x}_i + \frac{e}{c}\frac{dA_i}{dt} \\
&= m\ddot{x}_i + \frac{e}{c}\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}.
\end{aligned}
\tag{1.6}
$$

Assembling the results, we've got

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} \\
&= m\ddot{x}_i + \frac{e}{c}\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} - \frac{e}{c}v_j\frac{\partial A_j}{\partial x_i},
\end{aligned}
\tag{1.7}
$$

or

$$
\begin{aligned}
m\ddot{x}_i &= \frac{e}{c}v_j\frac{\partial A_j}{\partial x_i} - \frac{e}{c}\frac{\partial A_i}{\partial x_j}v_j \\
&= \frac{e}{c}v_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \\
&= \frac{e}{c}v_j B_k \epsilon_{ijk}.
\end{aligned}
\tag{1.8}
$$

In vector form that is

$$
m\ddot{\mathbf{x}} = \frac{e}{c}\mathbf{v} \times \mathbf{B}.
\tag{1.9}
$$

So, we get the magnetic term of the Lorentz force. Also note that this shows the Lagrangian (and the end result), was not in SI units. The $1/c$ term would have to be dropped for SI.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1