Peeter Joot
peeter.joot@gmail.com

A couple more bra-ket problems

Exercise 1.1 Operator matrix representation ([1] pr. 1.5)

1. Determine the matrix representation of $\left|\alpha\right\rangle\left\langle\beta\right|$ given a complete set of eigenvectors $\left|a^r\right\rangle$.
2. Verify with $\left|\alpha\right\rangle = \left|s_z = \hbar/2\right\rangle, \left|s_x = \hbar/2\right\rangle$.

Answer for Exercise 1.1

Part 1. Forming the matrix element

$$\begin{aligned}\left\langle a^r\right| \left(\left|\alpha\right\rangle\left\langle\beta\right| \right) \left|a^s\right\rangle &= \left\langle a^r|\alpha\right\rangle \left\langle\beta|a^s\right\rangle \\ &= \left\langle a^r|\alpha\right\rangle \left\langle a^s|\beta\right\rangle^*,\end{aligned} \tag{1.1}$$

the matrix representation is seen to be

$$\begin{aligned}\left|\alpha\right\rangle\left\langle\beta\right| &\sim \begin{bmatrix} \left\langle a^1\right| \left(\left|\alpha\right\rangle\left\langle\beta\right| \right) \left|a^1\right\rangle & \left\langle a^1\right| \left(\left|\alpha\right\rangle\left\langle\beta\right| \right) \left|a^2\right\rangle & \cdots \\ \left\langle a^2\right| \left(\left|\alpha\right\rangle\left\langle\beta\right| \right) \left|a^1\right\rangle & \left\langle a^2\right| \left(\left|\alpha\right\rangle\left\langle\beta\right| \right) \left|a^2\right\rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} \left\langle a^1|\alpha\right\rangle \left\langle a^1|\beta\right\rangle^* & \left\langle a^1|\alpha\right\rangle \left\langle a^2|\beta\right\rangle^* & \cdots \\ \left\langle a^2|\alpha\right\rangle \left\langle a^1|\beta\right\rangle^* & \left\langle a^2|\alpha\right\rangle \left\langle a^2|\beta\right\rangle^* & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.\end{aligned} \tag{1.2}$$

Part 2. First compute the spin-z representation of $\left|s_x = \hbar/2\right\rangle$.

$$\left(S_x - \hbar/2 I \right) \begin{bmatrix} a \\ b \end{bmatrix} = \left(\begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} - \begin{bmatrix} \hbar/2 & 0 \\ 0 & \hbar/2 \end{bmatrix} \right) = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{1.3}$$

so $\left|s_x = \hbar/2\right\rangle \propto (1, 1)$.
Normalized we have

$$\begin{aligned}\left|\alpha\right\rangle &= \left|s_z = \hbar/2\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \left|\beta\right\rangle &= \left|s_z = \hbar/2\right\rangle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned} \tag{1.4}$$

Using eq. (1.2) the matrix representation is

$$
\begin{aligned}
\left| \alpha \right\rangle \left\langle \beta \right| &\sim \begin{bmatrix} (1)(1/\sqrt{2})^* & (1)(1/\sqrt{2})^* \\ (0)(1/\sqrt{2})^* & (0)(1/\sqrt{2})^* \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{1.5}
$$

This can be confirmed with direct computation

$$
\begin{aligned}
\left| \alpha \right\rangle \left\langle \beta \right| &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{1.6}
$$

Exercise 1.2 eigenvalue of sum of kets ([1] *pr. 1.6*)

Given eigenkets $\left| i \right\rangle, \left| j \right\rangle$ of an operator A, what are the conditions that $\left| i \right\rangle + \left| j \right\rangle$ is also an eigenvector?

Answer for Exercise 1.2

Let $A \left| i \right\rangle = i \left| i \right\rangle, A \left| j \right\rangle = j \left| j \right\rangle$, and suppose that the sum is an eigenket. Then there must be a value a such that

$$
A \left(\left| i \right\rangle + \left| j \right\rangle \right) = a \left(\left| i \right\rangle + \left| j \right\rangle \right), \tag{1.7}
$$

so

$$
i \left| i \right\rangle + j \left| j \right\rangle = a \left(\left| i \right\rangle + \left| j \right\rangle \right). \tag{1.8}
$$

Operating with $\left\langle i \right|, \left\langle j \right|$ respectively, gives

$$
\begin{aligned}
i &= a \\
j &= a,
\end{aligned} \tag{1.9}
$$

so for the sum to be an eigenket, both of the corresponding energy eigenvalues must be identical (i.e. linear combinations of degenerate eigenkets are also eigenkets).

Exercise 1.3 Null operator ([1] *pr. 1.7*)

Given eigenkets $\left| a' \right\rangle$ of operator A

1. show that

$$
\prod_{a'} \left(A - a' \right) \tag{1.10}
$$

is the null operator.

2.

$$
\prod_{a'' \ne a'} \frac{\left(A - a'' \right)}{a' - a''} \tag{1.11}
$$

3. Illustrate using S_z for a spin 1/2 system.

Answer for Exercise 1.3

Part 1. Application of $|a\rangle$, the eigenket of A with eigenvalue a to any term $A - a'$ scales $|a\rangle$ by $a - a'$, so the product operating on $|a\rangle$ is

$$\prod_{a'} (A - a') |a\rangle = \prod_{a'} (a - a') |a\rangle . \tag{1.12}$$

Since $|a\rangle$ is one of the $\{|a'\rangle\}$ eigenkets of A, one of these terms must be zero.

Part 2. Again, consider the action of the operator on $|a\rangle$,

$$\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''} |a\rangle = \prod_{a'' \neq a'} \frac{(a - a'')}{a' - a''} |a\rangle . \tag{1.13}$$

If $|a\rangle = |a'\rangle$, then $\prod_{a'' \neq a'} \frac{(A-a'')}{a'-a''} |a\rangle = |a\rangle$, whereas if it does not, then it equals one of the a'' energy eigenvalues. This is a representation of the Kronecker delta function

$$\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''} |a\rangle \equiv \delta_{a',a} |a\rangle \tag{1.14}$$

Part 3. For operator S_z the eigenvalues are $\{\hbar/2, -\hbar/2\}$, so the null operator must be

$$\begin{aligned}
\prod_{a'} (A - a') &= \left(\frac{\hbar}{2}\right)^2 \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.15}$$

For the delta representation, consider the $|\pm\rangle$ states and their eigenvalue. The delta operators are

$$\prod_{a'' \neq \hbar/2} \frac{(A - a'')}{\hbar/2 - a''} = \frac{S_z - (-\hbar/2)I}{\hbar/2 - (-\hbar/2)} = \frac{1}{2}(\sigma_z + I) = \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{1.16}$$

$$\prod_{a'' \neq -\hbar/2} \frac{(A - a'')}{-\hbar/2 - a''} = \frac{S_z - (\hbar/2)I}{-\hbar/2 - \hbar/2} = \frac{1}{2}(\sigma_z - I) = \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \tag{1.17}$$

These clearly have the expected delta function property acting on kets $|+\rangle = (1, 0)$, $|-\rangle = (0, 1)$.

Exercise 1.4 **Spin half general normal** (*[1] pr. 1.9*)

Construct $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$, where $\hat{\mathbf{n}} = (\cos\alpha \sin\beta, \sin\alpha \sin\beta, \cos\beta)$ such that

$$\mathbf{S} \cdot \hat{\mathbf{n}} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \frac{\hbar}{2} |\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle , \tag{1.18}$$

Solve this as an eigenvalue problem.

Answer for Exercise 1.4

The spin operator for this direction is

$$\begin{aligned}\mathbf{S}\cdot\hat{\mathbf{n}} &= \frac{\hbar}{2}\sigma\cdot\hat{\mathbf{n}} \\ &= \frac{\hbar}{2}\left(\cos\alpha\sin\beta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \sin\alpha\sin\beta\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \cos\beta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \\ &= \frac{\hbar}{2}\begin{bmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta\end{bmatrix}.\end{aligned} \tag{1.19}$$

Observed that this is traceless and has a $-\hbar/2$ determinant like any of the x, y, z spin operators. Assuming that this has an $\hbar/2$ eigenvalue (to be verified later), the eigenvalue problem is

$$\begin{aligned}0 &= \mathbf{S}\cdot\hat{\mathbf{n}} - \hbar/2 I \\ &= \frac{\hbar}{2}\begin{bmatrix}\cos\beta - 1 & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta - 1\end{bmatrix} \\ &= \hbar\begin{bmatrix}-\sin^2\frac{\beta}{2} & e^{-i\alpha}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\ e^{i\alpha}\sin\frac{\beta}{2}\cos\frac{\beta}{2} & -\cos^2\frac{\beta}{2}\end{bmatrix}\end{aligned} \tag{1.20}$$

This has a zero determinant as expected, and the eigenvector (a, b) will satisfy

$$\begin{aligned}0 &= -\sin^2\frac{\beta}{2}a + e^{-i\alpha}\sin\frac{\beta}{2}\cos\frac{\beta}{2}b \\ &= \sin\frac{\beta}{2}\left(-\sin\frac{\beta}{2}a + e^{-i\alpha}b\cos\frac{\beta}{2}\right)\end{aligned} \tag{1.21}$$

$$\begin{bmatrix}a\\ b\end{bmatrix} \propto \begin{bmatrix}\cos\frac{\beta}{2}\\ e^{i\alpha}\sin\frac{\beta}{2}\end{bmatrix}. \tag{1.22}$$

This is appropriately normalized, so the ket for $\mathbf{S}\cdot\hat{\mathbf{n}}$ is

$$|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle. \tag{1.23}$$

Note that the other eigenvalue is

$$|\mathbf{S}\cdot\hat{\mathbf{n}};-\rangle = -\sin\frac{\beta}{2}|+\rangle + e^{i\alpha}\cos\frac{\beta}{2}|-\rangle. \tag{1.24}$$

It is straightforward to show that these are orthogonal and that this has the $-\hbar/2$ eigenvalue.

Exercise 1.5 Two state Hamiltonian ([1] *pr. 1.10*)

Solve the eigenproblem for

$$H = a\Big(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\Big) \tag{1.25}$$

Answer for Exercise 1.5

In matrix form the Hamiltonian is

$$H = a \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{1.26}$$

The eigenvalue problem is

$$\begin{aligned} 0 &= |H - \lambda I| \\ &= (a - \lambda)(-a - \lambda) - a^2 \\ &= (-a + \lambda)(a + \lambda) - a^2 \\ &= \lambda^2 - a^2 - a^2, \end{aligned} \tag{1.27}$$

or

$$\lambda = \pm\sqrt{2}a. \tag{1.28}$$

An eigenket proportional to (α, β) must satisfy

$$0 = (1 \mp \sqrt{2})\alpha + \beta, \tag{1.29}$$

so

$$|\pm\rangle \propto \begin{bmatrix} -1 \\ 1 \mp \sqrt{2} \end{bmatrix}, \tag{1.30}$$

or

$$\begin{aligned} |\pm\rangle &= \frac{1}{2(2 - \sqrt{2})} \begin{bmatrix} -1 \\ 1 \mp \sqrt{2} \end{bmatrix} \\ &= \frac{2 + \sqrt{2}}{4} \begin{bmatrix} -1 \\ 1 \mp \sqrt{2} \end{bmatrix}. \end{aligned} \tag{1.31}$$

That is

$$|\pm\rangle = \frac{2 + \sqrt{2}}{4} \left(- |1\rangle + (1 \mp \sqrt{2}) |2\rangle \right). \tag{1.32}$$

Exercise 1.6 Spin half probability and dispersion ([1] *pr. 1.12*)

A spin 1/2 system $\mathbf{S} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}} = \sin\gamma\hat{\mathbf{x}} + \cos\gamma\hat{\mathbf{z}}$, is in state with eigenvalue $\hbar/2$.

1. If S_x is measured. What is the probability of getting $+\hbar/2$?
2. Evaluate the dispersion in S_x, that is,

$$\left\langle (S_x - \langle S_x \rangle)^2 \right\rangle. \tag{1.33}$$

Answer for Exercise 1.6

Part 1. In matrix form the spin operator for the system is

$$\begin{aligned}\mathbf{S}\cdot\hat{\mathbf{n}} &= \frac{\hbar}{2}\left(\cos\gamma\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \sin\gamma\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\\ &= \frac{\hbar}{2}\begin{bmatrix}\cos\gamma & \sin\gamma\\ \sin\gamma & -\cos\gamma\end{bmatrix}\end{aligned} \tag{1.34}$$

An eigenket $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = (a,b)$ must satisfy

$$\begin{aligned}0 &= (\cos\gamma - 1)\,a + \sin\gamma b\\ &= \left(-2\sin^2\frac{\gamma}{2}\right)a + 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}b\\ &= -\sin\frac{\gamma}{2}a + \cos\frac{\gamma}{2}b,\end{aligned} \tag{1.35}$$

so the eigenstate is

$$|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \begin{bmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{bmatrix}. \tag{1.36}$$

Pick $|S_x;\pm\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \pm 1\end{bmatrix}$ as the basis for the S_x operator. Then, for the probability that the system will end up in the $+\hbar/2$ state of S_x, we have

$$\begin{aligned}P &= |\langle S_x;+|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle|^2\\ &= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}^\dagger\begin{bmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{bmatrix}\right|^2\\ &= \frac{1}{2}\left|\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{bmatrix}\right|^2\\ &= \frac{1}{2}\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)^2\\ &= \frac{1}{2}\left(1 + 2\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}\right)\\ &= \frac{1}{2}\left(1 + \sin\gamma\right).\end{aligned} \tag{1.37}$$

This is a reasonable seeming result, with $P \in [0,1]$. Some special values also further validate this

$$\begin{aligned}&\gamma = 0, |\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \begin{bmatrix}1\\ 0\end{bmatrix} = |S_z;+\rangle = \frac{1}{\sqrt{2}}|S_x;+\rangle + \frac{1}{\sqrt{2}}|S_x;-\rangle\\ &\gamma = \pi/2, |\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = |S_x;+\rangle\\ &\gamma = \pi, |\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \begin{bmatrix}0\\ 1\end{bmatrix} = |S_z;-\rangle = \frac{1}{\sqrt{2}}|S_x;+\rangle - \frac{1}{\sqrt{2}}|S_x;-\rangle,\end{aligned} \tag{1.38}$$

where we see that the probabilites are in proportion to the projection of the initial state onto the measured state $|S_x;+\rangle$.

Part 2. The S_x expectation is

$$\begin{aligned}
\langle S_x \rangle &= \frac{\hbar}{2} \begin{bmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{bmatrix} \\
&= \frac{\hbar}{2} \begin{bmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} \sin\frac{\gamma}{2} \\ \cos\frac{\gamma}{2} \end{bmatrix} \\
&= \frac{\hbar}{2} 2 \sin\frac{\gamma}{2} \cos\frac{\gamma}{2} \\
&= \frac{\hbar}{2} \sin\gamma.
\end{aligned} \tag{1.39}$$

Note that $S_x^2 = (\hbar/2)^2 I$, so

$$\begin{aligned}
\langle S_x^2 \rangle &= \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{bmatrix} \\
&= \left(\frac{\hbar}{2}\right)^2 \cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} \\
&= \left(\frac{\hbar}{2}\right)^2.
\end{aligned} \tag{1.40}$$

The dispersion is

$$\begin{aligned}
\left\langle (S_x - \langle S_x \rangle)^2 \right\rangle &= \langle S_x^2 \rangle - \langle S_x \rangle^2 \\
&= \left(\frac{\hbar}{2}\right)^2 \left(1 - \sin^2\gamma\right) \\
&= \left(\frac{\hbar}{2}\right)^2 \cos^2\gamma.
\end{aligned} \tag{1.41}$$

At $\gamma = \pi/2$ the dispersion is 0, which is expected since $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = |S_x; +\rangle$ at that point. Similarily, the dispersion is maximized at $\gamma = 0, \pi$ where the $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$ component in the $|S_x; +\rangle$ direction is minimized.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1, 1.2, 1.3, 1.4, 1.5, 1.6