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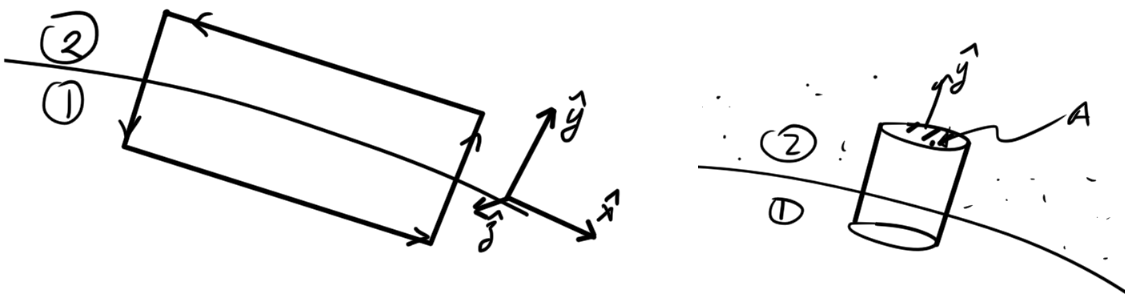
Tangential and normal field components

The integral forms of Maxwell's equations can be used to derive relations for the tangential and normal field components to the sources. These relations were mentioned in class. It's a little late, but lets go over the derivation. This isn't all review from first year electromagnetism since we are now using a magnetic source modifications of Maxwell's equations.

The derivation below follows that of [1] closely, but I am trying it myself to ensure that I understand the assumptions.

The two infinitesimally thin pillboxes of fig. 1.1 are used in the argument.



(a) (b)

Figure 1.1: Pillboxes for tangential and normal field relations

Maxwell's equations with both magnetic and electric sources are

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} - \boldsymbol{\mathcal{M}} \tag{1.1a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}} + \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} \tag{1.1b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{D}} = \rho_{\mathrm{e}} \tag{1.1c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = \rho_{\mathrm{m}}. \tag{1.1d}$$

After application of Stokes' and the divergence theorems Maxwell's equations have the integral form

$$\oint \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} = -\int d\mathbf{A} \cdot \left(\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \boldsymbol{\mathcal{M}} \right) \tag{1.2a}$$

$$\oint \boldsymbol{\mathcal{H}} \cdot d\mathbf{l} = \int d\mathbf{A} \cdot \left(\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{J}} \right) \tag{1.2b}$$

$$\int_{\partial V} \boldsymbol{\mathcal{D}} \cdot d\mathbf{A} = \int_V \rho_{\mathrm{e}} \, dV \tag{1.2c}$$

$$\int_{\partial V} \boldsymbol{\mathcal{B}} \cdot d\mathbf{A} = \int_V \rho_{\mathrm{m}} \, dV. \tag{1.2d}$$

1.1 Maxwell-Faraday equation

First consider one of the loop integrals, like eq. (1.2a). For an infinestismal loop, that integral is

$$\begin{aligned} \oint \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} &\approx \mathcal{E}_x^{(1)} \Delta x + \mathcal{E}^{(1)} \frac{\Delta y}{2} + \mathcal{E}^{(2)} \frac{\Delta y}{2} - \mathcal{E}_x^{(2)} \Delta x - \mathcal{E}^{(2)} \frac{\Delta y}{2} - \mathcal{E}^{(1)} \frac{\Delta y}{2} \\ &\approx \left(\mathcal{E}_x^{(1)} - \mathcal{E}_x^{(2)} \right) \Delta x + \frac{1}{2} \frac{\partial \mathcal{E}^{(2)}}{\partial x} \Delta x \Delta y + \frac{1}{2} \frac{\partial \mathcal{E}^{(1)}}{\partial x} \Delta x \Delta y. \end{aligned} \tag{1.3}$$

We let $\Delta y \rightarrow 0$ which kills off all but the first difference term.
The RHS of eq. (1.3) is approximately

$$-\int d\mathbf{A} \cdot \left(\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \boldsymbol{\mathcal{M}} \right) \approx -\Delta x \Delta y \left(\frac{\partial \mathcal{B}_z}{\partial t} + \mathcal{M}_z \right). \tag{1.4}$$

If the magnetic field contribution is assumed to be small in comparison to the magnetic current (i.e. infinite magnetic conductance), and if a linear magnetic current source of the form is also assumed

$$\boldsymbol{\mathcal{M}}_s = \lim_{\Delta y \rightarrow 0} \left(\boldsymbol{\mathcal{M}} \cdot \hat{\mathbf{z}} \right) \hat{\mathbf{z}} \Delta y, \tag{1.5}$$

then the Maxwell-Faraday equation takes the form

$$\left(\mathcal{E}_x^{(1)} - \mathcal{E}_x^{(2)} \right) \Delta x \approx -\Delta x \boldsymbol{\mathcal{M}}_s \cdot \hat{\mathbf{z}}. \tag{1.6}$$

While $\boldsymbol{\mathcal{M}}$ may have components that are not normal to the interface, the surface current need only have a normal component, since only that component contributes to the surface integral.
The coordinate expression of eq. (1.6) can be written as

$$\begin{aligned} -\boldsymbol{\mathcal{M}}_s \cdot \hat{\mathbf{z}} &= \left(\boldsymbol{\mathcal{E}}^{(1)} - \boldsymbol{\mathcal{E}}^{(2)} \right) \cdot \left(\hat{\mathbf{y}} \times \hat{\mathbf{z}} \right) \\ &= \left(\left(\boldsymbol{\mathcal{E}}^{(1)} - \boldsymbol{\mathcal{E}}^{(2)} \right) \times \hat{\mathbf{y}} \right) \cdot \hat{\mathbf{z}}. \end{aligned} \tag{1.7}$$

This is satisfied when

$$\boxed{\left(\boldsymbol{\mathcal{E}}^{(1)} - \boldsymbol{\mathcal{E}}^{(2)}\right) \times \hat{\mathbf{n}} = -\boldsymbol{\mathcal{M}}_s,} \tag{1.8}$$

where $\hat{\mathbf{n}}$ is the normal between the interfaces. I'd failed to understand when reading this derivation initially, how the $\boldsymbol{\mathcal{B}}$ contribution was killed off. i.e. If the vanishing area in the surface integral kills off the $\boldsymbol{\mathcal{B}}$ contribution, why do we have a $\boldsymbol{\mathcal{M}}$ contribution left. The key to this is understanding that this magnetic current is considered to be confined very closely to the surface getting larger as Δy gets smaller.

Also note that the units of $\boldsymbol{\mathcal{M}}_s$ are volts/meter like the electric field (not volts/squared-meter like $\boldsymbol{\mathcal{M}}$.)

1.2 Ampere's law

As above, assume a linear electric surface current density of the form

$$\boldsymbol{\mathcal{J}}_s = \lim_{\Delta y \rightarrow 0} (\boldsymbol{\mathcal{J}} \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}} \Delta y, \tag{1.9}$$

in units of amperes/meter (not amperes/meter-squared like $\boldsymbol{\mathcal{J}}$.)

To apply the arguments above to Ampere's law, only the sign needs to be adjusted

$$\boxed{\left(\boldsymbol{\mathcal{H}}^{(1)} - \boldsymbol{\mathcal{H}}^{(2)}\right) \times \hat{\mathbf{n}} = \boldsymbol{\mathcal{J}}_s.} \tag{1.10}$$

1.3 Gauss's law

Using the cylindrical pillbox surface with radius Δr, height Δy, and top and bottom surface areas $\Delta A = \pi\, (\Delta r)^2$, the LHS of Gauss's law eq. (1.2c) expands to

$$\begin{aligned} \int_{\partial V} \boldsymbol{\mathcal{D}} \cdot d\mathbf{A} &\approx \mathcal{D}_y^{(2)} \Delta A + \mathcal{D}_\rho^{(2)} 2\pi \Delta r \frac{\Delta y}{2} + \mathcal{D}_\rho^{(1)} 2\pi \Delta r \frac{\Delta y}{2} - \mathcal{D}_y^{(1)} \Delta A \\ &\approx \left(\mathcal{D}_y^{(2)} - \mathcal{D}_y^{(1)}\right) \Delta A. \end{aligned} \tag{1.11}$$

As with the Stokes integrals above it is assumed that the height is infinestimal with respect to the radial dimension. Letting that height $\Delta y \rightarrow 0$ kills off the radially directed contributions of the flux through the sidewalls.

The RHS expands to approximately

$$\int_V \rho_\mathrm{e}\, dV \approx \Delta A \Delta y \rho_\mathrm{e}. \tag{1.12}$$

Define a highly localized surface current density (coulombs/meter-squared) as

$$\sigma_\mathrm{e} = \lim_{\Delta y \rightarrow 0} \Delta y \rho_\mathrm{e}. \tag{1.13}$$

Equating eq. (1.12) with eq. (1.11) gives

$$\left(\mathcal{D}_y^{(2)} - \mathcal{D}_y^{(1)}\right) \Delta A = \Delta A \sigma_\mathrm{e}, \tag{1.14}$$

or

$$\boxed{\left(\boldsymbol{\mathcal{D}}^{(2)} - \boldsymbol{\mathcal{D}}^{(1)}\right) \cdot \hat{\mathbf{n}} = \sigma_\mathrm{e}.} \tag{1.15}$$

1.4 Gauss's law for magnetism

The same argument can be applied to the magnetic flux. Define a highly localized magnetic surface current density (webers/meter-squared) as

$$\sigma_{\mathrm{m}} = \lim_{\Delta y \to 0} \Delta y \rho_{\mathrm{m}}, \tag{1.16}$$

yielding the boundary relation

$$\boxed{\left(\boldsymbol{\mathcal{B}}^{(2)} - \boldsymbol{\mathcal{B}}^{(1)}\right) \cdot \hat{\mathbf{n}} = \sigma_{\mathrm{m}}.} \tag{1.17}$$

Bibliography

[1] Constantine A Balanis. *Advanced engineering electromagnetics*, volume 20, chapter Time-varying and time-harmonic electromagnetic fields. Wiley New York, 1989. 1