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Operator matrix element

1.1 Weird dreams

I woke up today having a dream still in my head from the night, but it was a strange one. I was expanding out the Dirac notation representation of an operator in matrix form, but the symbols in the kets were elaborate pictures of Disney princesses that I was drawing with forestry scenery in the background, including little bears. At the point that I woke up from the dream, I noticed that I'd gotten the proportion of the bears wrong in one of the pictures, and they looked like they were ready to eat one of the princess characters.

1.2 Guts

As a side effect of this weird dream I actually started thinking about matrix element representation of operators.

When forming the matrix element of an operator using Dirac notation the elements are of the form $\langle \text{row} | A | \text{column} \rangle$. I've gotten that mixed up a couple of times, so I thought it would be helpful to write this out explicitly for a 2×2 operator representation for clarity.

To start, consider a change of basis for a single matrix element from basis $\{ |q\rangle, |r\rangle \}$, to basis $\{ |a\rangle, |b\rangle \}$

$$\begin{aligned}
\langle q | A | r \rangle &= \langle q|a\rangle \langle a | A | r \rangle + \langle q|b\rangle \langle b | A | r \rangle \\
&= \langle q|a\rangle \langle a | A | a \rangle \langle a|r\rangle + \langle q|a\rangle \langle a | A | b \rangle \langle b|r\rangle \\
&+ \langle q|b\rangle \langle b | A | a \rangle \langle a|r\rangle + \langle q|b\rangle \langle b | A | b \rangle \langle b|r\rangle \\
&= \langle q|a\rangle \begin{bmatrix} \langle a | A | a \rangle & \langle a | A | b \rangle \end{bmatrix} \begin{bmatrix} \langle a|r\rangle \\ \langle b|r\rangle \end{bmatrix} + \langle q|b\rangle \begin{bmatrix} \langle b | A | a \rangle & \langle b | A | b \rangle \end{bmatrix} \begin{bmatrix} \langle a|r\rangle \\ \langle b|r\rangle \end{bmatrix} \\
&= \begin{bmatrix} \langle q|a\rangle & \langle q|b\rangle \end{bmatrix} \begin{bmatrix} \langle a | A | a \rangle & \langle a | A | b \rangle \\ \langle b | A | a \rangle & \langle b | A | b \rangle \end{bmatrix} \begin{bmatrix} \langle a|r\rangle \\ \langle b|r\rangle \end{bmatrix}.
\end{aligned} \tag{1.1}$$

Suppose the matrix representation of $|q\rangle, |r\rangle$ are respectively

$$
\begin{aligned}
|q\rangle &\sim \begin{bmatrix} \langle a|q\rangle \\ \langle b|q\rangle \end{bmatrix} \\
|r\rangle &\sim \begin{bmatrix} \langle a|r\rangle \\ \langle b|r\rangle \end{bmatrix},
\end{aligned}
\tag{1.2}
$$

then

$$
\begin{aligned}
\langle q| &\sim \begin{bmatrix} \langle a|q\rangle \\ \langle b|q\rangle \end{bmatrix}^\dagger \\
&= \begin{bmatrix} \langle q|a\rangle & \langle q|b\rangle \end{bmatrix}.
\end{aligned}
\tag{1.3}
$$

The matrix element is then

$$
\langle q| A |r\rangle \sim \langle q| \begin{bmatrix} \langle a| A |a\rangle & \langle a| A |b\rangle \\ \langle b| A |a\rangle & \langle b| A |b\rangle \end{bmatrix} |r\rangle , \tag{1.4}
$$

and the corresponding matrix representation of the operator is

$$
A \sim \begin{bmatrix} \langle a| A |a\rangle & \langle a| A |b\rangle \\ \langle b| A |a\rangle & \langle b| A |b\rangle \end{bmatrix}. \tag{1.5}
$$