Peeter Joot
peeter.joot@gmail.com

Partition function and ground state energy

Exercise 1.1 Partition function and ground state energy ([1] pr. 2.32)

Define the partition function as

$$Z = \int d^3 x' \, K(\mathbf{x}', t; \mathbf{x}', 0) \big|_{\beta = it/\hbar}, \tag{1.1}$$

Show that the ground state energy is given by

$$-\frac{1}{Z} \frac{\partial Z}{\partial \beta}, \qquad \beta \rightarrow \infty. \tag{1.2}$$

Answer for Exercise 1.1

The propagator evaluated at the same point is

$$\begin{aligned} K(\mathbf{x}', t; \mathbf{x}', 0) &= \sum_{a'} \langle \mathbf{x}' | a' \rangle \, |a' \rangle \, \mathbf{x}' \exp\left(-\frac{i E_{a'} t}{\hbar} \right) \\ &= \sum_{a'} \left| \langle \mathbf{x}' | a' \rangle \right|^2 \exp\left(-\frac{i E_{a'} t}{\hbar} \right) \\ &= \sum_{a'} \left| \langle \mathbf{x}' | a' \rangle \right|^2 \exp\left(-E_{a'} \beta \right). \end{aligned} \tag{1.3}$$

The derivative is

$$\frac{\partial Z}{\partial \beta} = - \int d^3 x' \sum_{a'} E_{a'} \left| \langle \mathbf{x}' | a' \rangle \right|^2 \exp\left(-E_{a'} \beta \right). \tag{1.4}$$

In the $\beta \rightarrow \infty$ this sum will be dominated by the term with the lowest value of $E_{a'}$. Suppose that state is $a' = 0$, then

$$\begin{aligned} \lim_{\beta \rightarrow \infty} -\frac{1}{Z} \frac{\partial Z}{\partial \beta} &= \frac{\int d^3 x' E_0 |\langle \mathbf{x}' | 0 \rangle|^2 \exp\left(-E_0 \beta \right)}{\int d^3 x' |\langle \mathbf{x}' | 0 \rangle|^2 \exp\left(-E_0 \beta \right)} \\ &= E_0. \end{aligned} \tag{1.5}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1