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An observation about the geometry of Pauli x,y matrices

1.1 Motivation

The conventional form for the Pauli matrices is

$$\begin{aligned}\sigma_x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \\ \sigma_z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned} \tag{1.1}$$

In [1] these forms are derived based on the commutation relations

$$\left[\sigma_r, \sigma_s\right] = 2i\epsilon_{rst}\sigma_t, \tag{1.2}$$

by defining raising and lowering operators $\sigma_\pm = \sigma_x \pm i\sigma_y$ and figuring out what form the matrix must take. I noticed an interesting geometrical relation hiding in that derivation if σ_+ is not assumed to be real.

1.2 Derivation

For completeness, I'll repeat the argument of [1], which builds on the commutation relations of the raising and lowering operators. Those are

$$\begin{aligned}\left[\sigma_z, \sigma_\pm\right] &= \sigma_z \left(\sigma_x \pm i\sigma_y\right) - \left(\sigma_x \pm i\sigma_y\right) \sigma_z \\ &= \left[\sigma_z, \sigma_x\right] \pm i \left[\sigma_z, \sigma_y\right] \\ &= 2i\sigma_y \pm i(-2i)\sigma_x \\ &= \pm 2 \left(\sigma_x \pm i\sigma_y\right) \\ &= \pm 2\sigma_\pm,\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
[\sigma_+, \sigma_-] &= \left(\sigma_x + i\sigma_y\right)\left(\sigma_x - i\sigma_y\right) - \left(\sigma_x - i\sigma_y\right)\left(\sigma_x + i\sigma_y\right) \\
&= -i\sigma_x\sigma_y + i\sigma_y\sigma_x - i\sigma_x\sigma_y + i\sigma_y\sigma_x \\
&= 2i\left[\sigma_y, \sigma_x\right] \\
&= 2i(-2i)\sigma_z \\
&= 4\sigma_z
\end{aligned} \tag{1.4}$$

From these a matrix representation containing unknown values can be assumed. Let

$$\sigma_+ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.5}$$

The commutator with σ_z can be computed

$$\begin{aligned}
[\sigma_z, \sigma_+] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ -c & -d \end{bmatrix} - \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \\
&= 2\begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}
\end{aligned} \tag{1.6}$$

Now compare this with eq. (1.3)

$$\begin{aligned}
2\begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} &= 2\sigma_+ \\
&= 2\begin{bmatrix} a & b \\ d & d \end{bmatrix}.
\end{aligned} \tag{1.7}$$

This shows that $a = 0$, and $d = 0$. Similarly the σ_z commutator with the lowering operator is

$$\begin{aligned}
[\sigma_z, \sigma_-] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -c^* \\ b^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & -c^* \\ b^* & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -c^* \\ -b^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix} \\
&= -2\begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix}
\end{aligned} \tag{1.8}$$

Again comparing to eq. (1.3), we have

$$\begin{aligned}
-2\begin{bmatrix} 0 & c^* \\ b^* & 0 \end{bmatrix} &= -2\sigma_- \\
&= -2\begin{bmatrix} 0 & -c^* \\ b^* & 0 \end{bmatrix},
\end{aligned} \tag{1.9}$$

so $c = 0$. Computing the commutator of the raising and lowering operators fixes b

$$
\begin{aligned}
[\sigma_+, \sigma_-] &= \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ b^* & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} |b|^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 00 & -|b|^2 \end{bmatrix} \\
&= |b|^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= |b|^2 \sigma_z.
\end{aligned}
\tag{1.10}
$$

From eq. (1.4) it must be that $|b|^2 = 4$, so the most general form of the raising operator is

$$
\sigma_+ = 2 \begin{bmatrix} 0 & e^{i\phi} \\ 0 & 0 \end{bmatrix}.
\tag{1.11}
$$

1.3 Observation

The conventional choice is to set $\phi = 0$, but I found it interesting to see the form of σ_x, σ_y without that choice. That is

$$
\begin{aligned}
\sigma_x &= \frac{1}{2} (\sigma_+ + \sigma_-) \\
&= \begin{bmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{bmatrix}
\end{aligned}
\tag{1.12}
$$

$$
\begin{aligned}
\sigma_y &= \frac{1}{2i} (\sigma_+ - \sigma_-) \\
&= \begin{bmatrix} 0 & -ie^{i\phi} \\ -ie^{-i\phi} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & e^{i(\phi - \pi/2)} \\ e^{-i(\phi - \pi/2)} & 0 \end{bmatrix}.
\end{aligned}
\tag{1.13}
$$

Notice that the Pauli matrices σ_x and σ_y actually both have the same form as σ_x, but the phase of the complex argument of each differs by 90°. That 90° separation isn't obvious in the standard form eq. (1.1).

It's a small detail, but I thought it was kind of cool that the orthogonality of these matrix unit vector representations is built directly into the structure of their matrix representations.

Bibliography

[1] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 1.1, 1.2