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Polarization review

It seems worthwhile to review how a generally polarized field phasor leads to linear, circular, and elliptic geometries.

The most general field polarized in the x, y plane has the form

$$\begin{aligned}\mathbf{E} &= \left(\hat{\mathbf{x}} a e^{j\alpha} + \hat{\mathbf{y}} b e^{j\beta} \right) e^{j(\omega t - k z)} \\ &= \left(\hat{\mathbf{x}} a e^{j(\alpha - \beta)/2} + \hat{\mathbf{y}} b e^{j(\beta - \alpha)/2} \right) e^{j(\omega t - k z + (\alpha + \beta)/2)}.\end{aligned} \tag{1.1}$$

Knowing to factor out the average phase angle above is only because I tried initially without that and things got ugly and messy. I guessed this would help (it does).

Let $\boldsymbol{\mathcal{E}} = \operatorname{Re} \mathbf{E} = \hat{\mathbf{x}} x + \hat{\mathbf{y}} y$, $\theta = \omega t + (\alpha + \beta)/2$, and $\phi = (\alpha - \beta)/2$, so that

$$\mathbf{E} = \left(\hat{\mathbf{x}} a e^{j\phi} + \hat{\mathbf{y}} b e^{-j\phi} \right) e^{j\theta}. \tag{1.2}$$

The coordinates can now be read off

$$\frac{x}{a} = \cos\phi \cos\theta - \sin\phi \sin\theta \tag{1.3a}$$

$$\frac{y}{b} = \cos\phi \cos\theta + \sin\phi \sin\theta, \tag{1.3b}$$

or in matrix form

$$\begin{bmatrix} x/a \\ y/b \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{1.4}$$

The goal is to eliminate all the θ (i.e. time dependence), converting the parametric relationship into a conic form. Assuming that neither $\cos\theta$, nor $\sin\theta$ are zero for now (those are special cases and lead to linear polarization), inverting the matrix will allow the θ dependence to be eliminated

$$\frac{1}{\sin(2\phi)} \begin{bmatrix} \sin\phi & \sin\phi \\ -\cos\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x/a \\ y/b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{1.5}$$

Squaring and summing both rows of these equation gives

$$\begin{aligned}
1 &= \frac{1}{\sin^2(2\phi)} \left(\sin^2\phi \left(\frac{x}{a} + \frac{y}{b} \right)^2 + \cos^2\phi \left(-\frac{x}{a} + \frac{y}{b} \right)^2 \right) \\
&= \frac{1}{\sin^2(2\phi)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\frac{xy}{ab} \left(\sin^2\phi - \cos^2\phi \right) \right) \\
&= \frac{1}{\sin^2(2\phi)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab} \cos(2\phi) \right)
\end{aligned} \tag{1.6}$$

Time to summarize and handle the special cases.

1. To have $\cos\phi = 0$, the phase angles must satisfy $\alpha - \beta = (1 + 2k)\pi$, $k \in \mathbb{Z}$.

   For this case eq. (1.3) reduces to

   $$-\frac{x}{a} = \frac{y}{b}, \tag{1.7}$$

   which is just a line.

   Example. Let $\alpha = 0, \beta = -\pi$, so that the phasor has the value

   $$\mathbf{E} = (\hat{\mathbf{x}}a - \hat{\mathbf{y}}b)\, e^{j\omega t} \tag{1.8}$$

2. For have $\sin\phi = 0$, the phase angles must satisfy $\alpha - \beta = 2\pi k$, $k \in \mathbb{Z}$.

   For this case eq. (1.3) reduces to

   $$\frac{x}{a} = \frac{y}{b}, \tag{1.9}$$

   also just a line.

   Example. Let $\alpha = \beta = 0$, so that the phasor has the value

   $$\mathbf{E} = (\hat{\mathbf{x}}a + \hat{\mathbf{y}}b)\, e^{j\omega t} \tag{1.10}$$

3. Last is the circular and elliptically polarized case. The system is clearly elliptically polarized if $\cos(2\phi) = 0$, or $\alpha - \beta = (\pi/2)(1 + 2k), k \in \mathbb{Z}$. When that is the case and $a = b$ also holds, the ellipse is a circle.

   When the $\cos(2\phi) = 0$ condition does not hold, a rotation of coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{1.11}$$

where

$$\mu = \frac{1}{2}\tan^{-1}\left(\frac{2\cos(\alpha - \beta)}{b - a}\right) \tag{1.12}$$

puts the trajectory into a standard (but messy) conic form

$$1 = \frac{u^2}{ab}\left(\frac{b}{a}\cos^2\mu + \frac{a}{b}\sin^2\mu + \frac{1}{2}\sin\left(2\mu + \alpha - \beta\right)\right) + \frac{v^2}{ab}\left(\frac{b}{a}\sin^2\mu + \frac{a}{b}\cos^2\mu - \frac{1}{2}\sin\left(2\mu + \alpha - \beta\right)\right) \tag{1.13}$$

It isn't obvious to me that the factors of the u^2, v^2 terms are necessarily positive, which is required for the conic to be an ellipse and not a hyperbola.

Circular polarization example. With $a = b = E_0$, $\alpha = 0$, $\beta = \pm\pi/2$, all the circular polarization conditions are met, leaving the phasor with values

$$\mathbf{E} = E_0\left(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}}\right)e^{j\omega t} \tag{1.14}$$