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Heisenberg picture position commutator

Exercise 1.1 Heisenberg picture position commutator ([1] *pr. 2.5*)

Evaluate

$$[x(t), x(0)] , \tag{1.1}$$

for a Heisenberg picture operator $x(t)$ for a free particle.

Answer for Exercise 1.1

The free particle Hamiltonian is

$$H = \frac{p^2}{2m}, \tag{1.2}$$

so the time evolution operator is

$$U(t) = e^{-ip^2 t/(2m\hbar)}. \tag{1.3}$$

The Heisenberg picture position operator is

$$
\begin{aligned}
x^{\mathrm{H}} &= U^\dagger x U \\
&= e^{ip^2t/(2m\hbar)} x e^{-ip^2t/(2m\hbar)} \\
&= \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{ip^2t}{2m\hbar}\right)^k x e^{-ip^2t/(2m\hbar)} \\
&= \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{it}{2m\hbar}\right)^k p^{2k} x e^{-ip^2t/(2m\hbar)} \\
&= \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{it}{2m\hbar}\right)^k \left(\left[p^{2k}, x\right] + x p^{2k}\right) e^{-ip^2t/(2m\hbar)} \\
&= x + \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{it}{2m\hbar}\right)^k \left[p^{2k}, x\right] e^{-ip^2t/(2m\hbar)} \\
&= x + \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{it}{2m\hbar}\right)^k \left(-i\hbar \frac{\partial p^{2k}}{\partial p}\right) e^{-ip^2t/(2m\hbar)} \\
&= x + \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{it}{2m\hbar}\right)^k \left(-i\hbar 2k p^{2k-1}\right) e^{-ip^2t/(2m\hbar)} \\
&= x + -2i\hbar p \frac{it}{2m\hbar} \sum_{k=1}^\infty \frac{1}{(k-1)!} \left(\frac{it}{2m\hbar}\right)^{k-1} p^{2(k-1)} e^{-ip^2t/(2m\hbar)} \\
&= x + t\frac{p}{m}.
\end{aligned}
\tag{1.4}
$$

This has the structure of a classical free particle $x(t) = x + vt$, but in this case x, p are operators. The evolved position commutator is

$$
\begin{aligned}
[x(t), x(0)] &= \left[x + tp/m, x\right] \\
&= \frac{t}{m} \left[p, x\right] \\
&= -i\hbar \frac{t}{m}.
\end{aligned}
\tag{1.5}
$$

Compare this to the classical Poisson bracket

$$
\begin{aligned}
[x(t), x(0)]_{\text{classical}} &= \frac{\partial}{\partial x} \left(x + pt/m\right) \frac{\partial x}{\partial p} - \frac{\partial}{\partial p} \left(x + pt/m\right) \frac{\partial x}{\partial x} \\
&= -\frac{t}{m}.
\end{aligned}
\tag{1.6}
$$

This has the expected relation $\left[x(t), x(0)\right] = i\hbar \left[x(t), x(0)\right]_{\text{classical}}$.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1