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PHY1520H Graduate Quantum Mechanics. Lecture 11: Symmetries in QM. Taught by Prof. Arun Paramekanti

Disclaimer Peeter's lecture notes from class. These may be incoherent and rough.

These are notes for the UofT course PHY1520, Graduate Quantum Mechanics, taught by Prof. Paramekanti, covering ch. 4 [1] content.

Symmetry in classical mechanics In a classical context considering a Hamiltonian

$$H(q_i, p_i), \tag{1.1}$$

a symmetry means that certain q_i don't appear. In that case the rate of change of one of the generalized momenta is zero

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} = 0, \tag{1.2}$$

so p_k is a constant of motion. This simplifies the problem by reducing the number of degrees of freedom. Another aspect of such a symmetry is that it relates trajectories. For example, assuming a rotational symmetry as in fig. 1.1.

the trajectory of a particle after rotation is related by rotation to the trajectory of the unrotated particle.

Symmetry in quantum mechanics Suppose that we have a symmetry operation that takes states from

$$\left|\psi\right\rangle \rightarrow \left|U\psi\right\rangle \tag{1.3}$$

$$\left|\phi\right\rangle \rightarrow \left|U\phi\right\rangle, \tag{1.4}$$

we expect that

$$\left|\left\langle\psi|\phi\right\rangle\right|^2 = \left|\left\langle U\psi|U\phi\right\rangle\right|^2. \tag{1.5}$$

This won't hold true for a general operator. Two cases where this does hold true is when

- $\left\langle\psi|\phi\right\rangle = \left\langle U\psi|U\phi\right\rangle$. Here U is unitary, and the equivalence follows from

$$\left\langle U\psi|U\phi\right\rangle = \left\langle\psi\right| U^\dagger U\phi = \left\langle\psi\right| 1\phi = \left\langle\psi|\phi\right\rangle. \tag{1.6}$$

- $\left\langle\psi|\phi\right\rangle = \left\langle U\psi|U\phi\right\rangle^*$. Here U is anti-unitary.

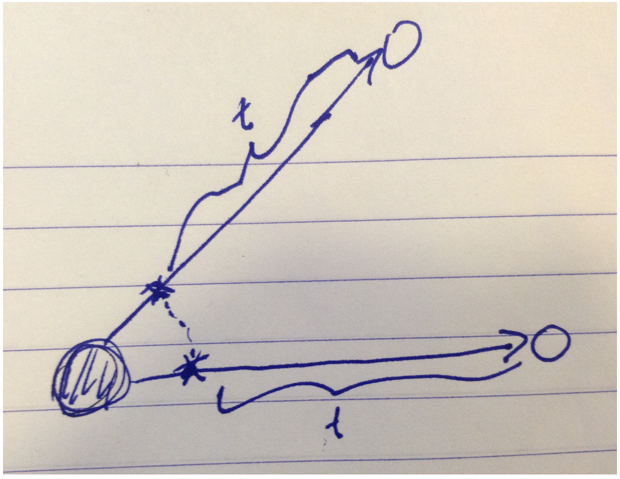

Figure 1.1: Trajectory under rotational symmetry

Unitary case If an “observable” is not changed by a unitary operation representing a symmetry we must have

$$
\begin{aligned}
\langle\psi|\,\hat{A}\,|\psi\rangle &\rightarrow \langle U\psi|\,\hat{A}\,|U\psi\rangle \\
&= \langle\psi|\,U^\dagger\hat{A}U\,|\psi\rangle\,,
\end{aligned}
\tag{1.7}
$$

so

$$
U^\dagger\hat{A}U = \hat{A},
\tag{1.8}
$$

or

$$
\boxed{\hat{A}U = U\hat{A}.}
\tag{1.9}
$$

An observable that is unchanged by a unitary symmetry commutes $\left[\hat{A}, U\right]$ with the operator U for that transformation.

Symmetries of the Hamiltonian Given

$$
[H, U] = 0,
\tag{1.10}
$$

H is invariant.
Given

$$
H\,|\phi_n\rangle = \epsilon_n\,|\phi_n\rangle\,.
\tag{1.11}
$$

$$
\begin{aligned}
UH\,|\phi_n\rangle &= HU\,|\phi_n\rangle \\
&= \epsilon_n U\,|\phi_n\rangle
\end{aligned}
\tag{1.12}
$$

Such a state

$$
|\psi_n\rangle = U\,|\phi_n\rangle
\tag{1.13}
$$

is also an eigenstate with the same energy.
Suppose this process is repeated, finding other states

$$U |\psi_n\rangle = |\chi_n\rangle \tag{1.14}$$

$$U |\chi_n\rangle = |\alpha_n\rangle \tag{1.15}$$

Because such a transformation only generates states with the initial energy, this process cannot continue forever. At some point this process will enumerate a fixed size set of states. These states can be orthonormalized.

We can say that symmetry operations are generators of a group. For a set of symmetry operations we can

- Form products that lie in a closed set

$$U_1 U_2 = U_3 \tag{1.16}$$

- can define an inverse

$$U \leftrightarrow U^{-1}. \tag{1.17}$$

- obeys associative rules for multiplication

$$U_1(U_2 U_3) = (U_1 U_2) U_3. \tag{1.18}$$

- has an identity operation.

When H has a symmetry, then degenerate eigenstates form irreducible representations (which cannot be further block diagonalized).

Some simple examples

Example 1.1: Inversion.

Given a state and a parity operation $\hat{\Pi}$, with the transformation

$$|\psi\rangle \rightarrow \hat{\Pi} |\psi\rangle \tag{1.19}$$

In one dimension, the parity operation is just inversion. In two dimensions, this is a set of flipping operations on two axes fig. 1.2.

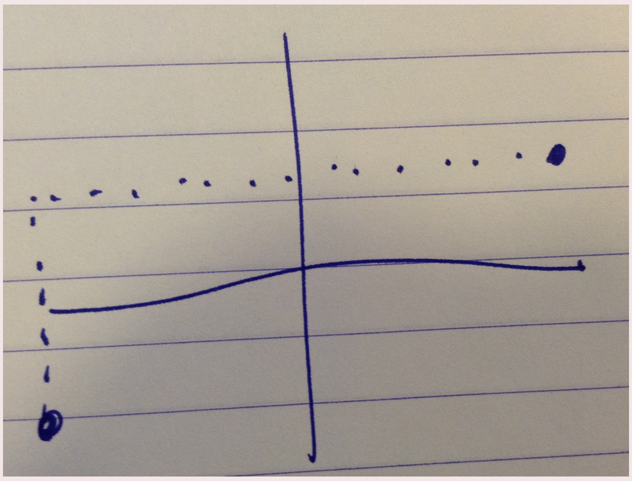

Figure 1.2: 2D parity operation

The operational effects of this operator are

$$\begin{aligned} \hat{x} &\rightarrow -\hat{x} \\ \hat{p} &\rightarrow -\hat{p}. \end{aligned} \tag{1.20}$$

Acting again with the parity operator produces the original value, so it is its own inverse, and $\hat{\Pi}^\dagger = \hat{\Pi} = \hat{\Pi}^{-1}$. In an expectation value

$$\left\langle \hat{\Pi}\psi \right| \hat{x} \left| \hat{\Pi}\psi \right\rangle = - \left\langle \psi \right| \hat{x} \left| \psi \right\rangle . \tag{1.21}$$

This means that

$$\hat{\Pi}^\dagger \hat{x} \hat{\Pi} = -\hat{x}, \tag{1.22}$$

or

$$\hat{x}\hat{\Pi} = -\hat{\Pi}\hat{x}, \tag{1.23}$$

$$\begin{aligned} \hat{x}\hat{\Pi} \left| x_0 \right\rangle &= -\hat{\Pi}\hat{x} \left| x_0 \right\rangle \\ &= -\hat{\Pi} x_0 \left| x_0 \right\rangle \\ &= -x_0 \hat{\Pi} \left| x_0 \right\rangle \end{aligned} \tag{1.24}$$

so

$$\hat{\Pi} \left| x_0 \right\rangle = \left| -x_0 \right\rangle . \tag{1.25}$$

Acting on a wave function

$$\begin{aligned}\langle x|\,\hat{\Pi}\,|\psi\rangle &= \langle -x|\psi\rangle \\ &= \psi(-x).\end{aligned} \tag{1.26}$$

What does this mean for eigenfunctions. Eigenfunctions are supposed to form irreducible representations of the group. The group has just two elements

$$\left\{1, \hat{\Pi}\right\}, \tag{1.27}$$

where $\hat{\Pi}^2 = 1$.

Suppose we have a Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{1.28}$$

where $V(\hat{x})$ is even ($\left[V(\hat{x}), \hat{\Pi}\right] = 0$). The squared momentum commutes with the parity operator

$$\begin{aligned}\left[\hat{p}^2, \hat{\Pi}\right] &= \hat{p}^2\hat{\Pi} - \hat{\Pi}\hat{p}^2 \\ &= \hat{p}^2\hat{\Pi} - (\hat{\Pi}\hat{p})\hat{p} \\ &= \hat{p}^2\hat{\Pi} - (-\hat{p}\hat{\Pi})\hat{p} \\ &= \hat{p}^2\hat{\Pi} + \hat{p}(-\hat{p}\hat{\Pi}) \\ &= 0.\end{aligned} \tag{1.29}$$

Only two functions are possible in the symmetry set $\left\{\Psi(x), \hat{\Pi}\Psi(x)\right\}$, since

$$\begin{aligned}\hat{\Pi}^2\Psi(x) &= \hat{\Pi}\Psi(-x) \\ &= \Psi(x).\end{aligned} \tag{1.30}$$

This symmetry severely restricts the possible solutions, making it so there can be only one dimensional forms of this problem with solutions that are either even or odd respectively

$$\begin{aligned}\phi_e(x) &= \psi(x) + \psi(-x) \\ \phi_o(x) &= \psi(x) - \psi(-x).\end{aligned} \tag{1.31}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1