Peeter Joot
peeter.joot@gmail.com

Reciprocity theorem in Geometric Algebra

The reciprocity theorem involves a Poynting like antisymmetric difference of the following form

$$\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)}. \tag{1.1}$$

This smells like something that can probably be related to a combined electromagnetic field multivectors in some sort of structured fashion. Guessing that this is related to the antisymmetic sum of two electromagnetic field multivectors turns out to be correct. Let

$$F^{(a)} = \mathbf{E}^{(a)} + Ic\mathbf{B}^{(a)} \tag{1.2a}$$

$$F^{(b)} = \mathbf{E}^{(b)} + Ic\mathbf{B}^{(b)}. \tag{1.2b}$$

Now form the antisymmetic sum

$$\begin{aligned}
\frac{1}{2}\left(F^{(a)}F^{(b)} - F^{(b)}F^{(a)}\right) &= \frac{1}{2}\left(\mathbf{E}^{(a)} + Ic\mathbf{B}^{(a)}\right)\left(\mathbf{E}^{(b)} + Ic\mathbf{B}^{(b)}\right) - \frac{1}{2}\left(\mathbf{E}^{(b)} + Ic\mathbf{B}^{(b)}\right)\left(\mathbf{E}^{(a)} + Ic\mathbf{B}^{(a)}\right) \\
&= \frac{1}{2}\left(\mathbf{E}^{(a)}\mathbf{E}^{(b)} - \mathbf{E}^{(b)}\mathbf{E}^{(a)}\right) + \frac{Ic}{2}\left(\mathbf{E}^{(a)}\mathbf{B}^{(b)}\right. \\
&\quad \left.- \mathbf{B}^{(b)}\mathbf{E}^{(a)}\right) + \frac{Ic}{2}\left(\mathbf{B}^{(a)}\mathbf{E}^{(b)} - \mathbf{E}^{(b)}\mathbf{B}^{(a)}\right) + \frac{c^2}{2}\left(\mathbf{B}^{(b)}\mathbf{B}^{(a)} - \mathbf{B}^{(a)}\mathbf{B}^{(b)}\right) \\
&= \mathbf{E}^{(a)} \wedge \mathbf{E}^{(b)} + c^2\left(\mathbf{B}^{(b)} \wedge \mathbf{B}^{(a)}\right) + Ic\left(\mathbf{E}^{(a)} \wedge \mathbf{B}^{(b)} + \mathbf{B}^{(a)} \wedge \mathbf{E}^{(b)}\right) \\
&= I\mathbf{E}^{(a)} \times \mathbf{E}^{(b)} + c^2 I\left(\mathbf{B}^{(b)} \times \mathbf{B}^{(a)}\right) - c\left(\mathbf{E}^{(a)} \times \mathbf{B}^{(b)} + \mathbf{B}^{(a)} \times \mathbf{E}^{(b)}\right)
\end{aligned} \tag{1.3}$$

This has two components, the first is a bivector (pseudoscalar times vector) that includes all the non-mixed products, and the second is a vector that includes all the mixed terms. We can therefore write the antisymmetic difference of the reciprocity theorem by extracting just the grade one terms of the antisymmetic sum of the combined electromagnetic field

$$\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} = -\frac{1}{2c\mu_0}\left\langle \left(F^{(a)}F^{(b)} - F^{(b)}F^{(a)}\right)\right\rangle_1. \tag{1.4}$$

Observing that the antisymmetrization used in the reciprocity theorem is only one portion of the larger electromagnetic field antisymmetrization, introduces two new questions

1. How would the reciprocity theorem be derived directly in terms of $F^{(a)}F^{(b)} - F^{(b)}F^{(a)}$?
2. What is the significance of the other portion of this antisymmetrization $\mathbf{E}^{(a)} \times \mathbf{E}^{(b)} - c^2\mu_0^2 \left(\mathbf{H}^{(a)} \times \mathbf{H}^{(b)}\right)$?