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Cascading Stern-Gerlach

Exercise 1.1 Cascading Stern-Gerlach ([1] *pr. 1.13*)

Three Stern-Gerlach type measurements are performed, the first that prepares the state in a $|S_z;+\rangle$ state, the next in a $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ state where $\hat{\mathbf{n}} = \cos\beta\hat{\mathbf{z}} + \sin\beta\hat{\mathbf{x}}$, and the last performing a S_z $\hbar/2$ state measurement, as illustrated in fig. 1.1.

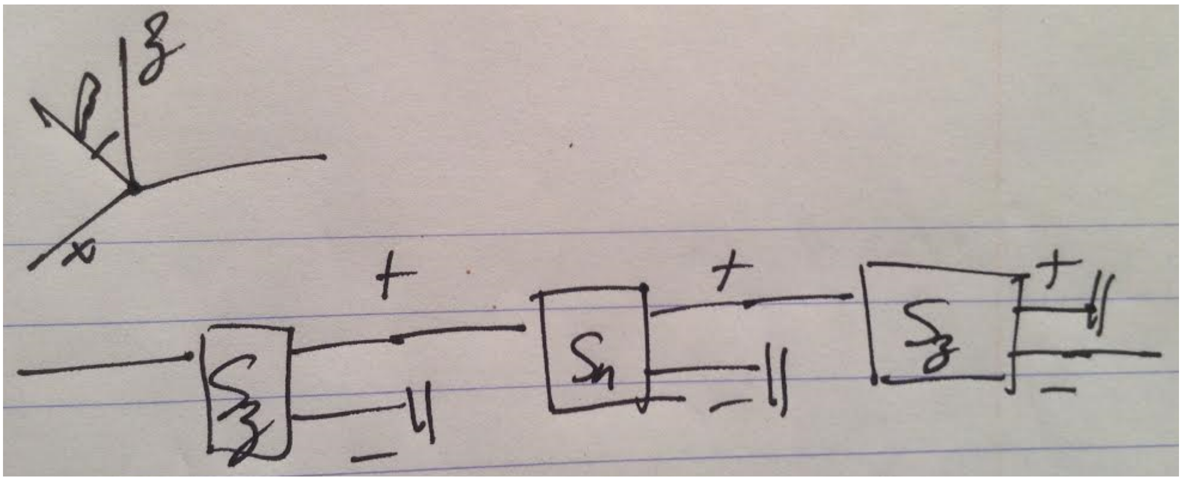

Figure 1.1: Cascaded Stern-Gerlach type measurements.

What is the intensity of the final $s_z = -\hbar/2$ beam? What is the orientation for the second measuring apparatus to maximize the intensity of this beam?

Answer for Exercise 1.1

The spin operator for the second apparatus is

$$\begin{aligned}\mathbf{S}\cdot\hat{\mathbf{n}} &= \frac{\hbar}{2}\left(\sin\beta\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \cos\beta\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right)\\&= \frac{\hbar}{2}\begin{bmatrix}\cos\beta & \sin\beta\\\sin\beta & -\cos\beta\end{bmatrix}.\end{aligned} \tag{1.1}$$

The intensity of the final $|S_z; -\rangle$ beam is

$$P = \left| \langle - | \mathbf{S} \cdot \hat{\mathbf{n}}; + \rangle \langle \mathbf{S} \cdot \hat{\mathbf{n}}; + | + \rangle \right|^2, \tag{1.2}$$

(i.e. the second apparatus applies a projection operator $|\mathbf{S} \cdot \hat{\mathbf{n}}; + \rangle \langle \mathbf{S} \cdot \hat{\mathbf{n}}; +|$ to the initial $|+\rangle$ state, and then the $|-\rangle$ states are selected out of that.

The $\mathbf{S} \cdot \hat{\mathbf{n}}$ eigenket is found to be

$$|\mathbf{S} \cdot \hat{\mathbf{n}}; + \rangle = \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix}, \tag{1.3}$$

so

$$\begin{aligned} P &= \left| \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|^2 \\ &= \left| \cos\frac{\beta}{2} \sin\frac{\beta}{2} \right|^2 \\ &= \left| \frac{1}{2} \sin\beta \right|^2 \\ &= \frac{1}{4} \sin^2\beta. \end{aligned} \tag{1.4}$$

This is maximized when $\beta = \pi/2$, or $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. At this angle the state leaving the second apparatus is

$$\begin{aligned} \begin{bmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} |+\rangle + \frac{1}{2} |-\rangle, \end{aligned} \tag{1.5}$$

so the state after filtering the $|-\rangle$ states is $\frac{1}{2} |-\rangle$ with intensity (probability density) of $1/4$ relative to a unit normalize input $|+\rangle$ state to the $\mathbf{S} \cdot \hat{\mathbf{n}}$ apparatus.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1