Peeter Joot
peeterjoot@protonmail.com

1D SHO linear superposition that maximizes expectation

Exercise 1.1 **1D SHO linear superposition that maximizes expectation ([1] *pr. 2.17*)**

For a 1D SHO

1. Construct a linear combination of $\left|0\right\rangle, \left|1\right\rangle$ that maximizes $\left\langle x \right\rangle$ without using wave functions.
2. How does this state evolve with time?
3. Evaluate $\left\langle x \right\rangle$ using the Schrödinger picture.
4. Evaluate $\left\langle x \right\rangle$ using the Heisenberg picture.
5. Evaluate $\left\langle (\Delta x)^2 \right\rangle$.

Answer for Exercise 1.1

Part 1. Forming

$$\left|\psi\right\rangle = \frac{\left|0\right\rangle + \sigma \left|1\right\rangle}{\sqrt{1 + \left|\sigma\right|^2}} \tag{1.1}$$

the position expectation is

$$\left\langle \psi \right| x \left| \psi \right\rangle = \frac{1}{1 + \left|\sigma\right|^2} \left(\left\langle 0 \right| + \sigma^* \left\langle 1 \right| \right) \frac{x_0}{\sqrt{2}} \left(a^\dagger + a \right) \left(\left|0\right\rangle + \sigma \left|1\right\rangle \right). \tag{1.2}$$

Evaluating the action of the operators on the kets, we've got

$$\left(a^\dagger + a \right) \left(\left|0\right\rangle + \sigma \left|1\right\rangle \right) = \left|1\right\rangle + \sqrt{2}\sigma \left|2\right\rangle + \sigma \left|0\right\rangle. \tag{1.3}$$

The $\left|2\right\rangle$ term is killed by the bras, leaving

$$\begin{aligned}
\left\langle x \right\rangle &= \frac{1}{1 + \left|\sigma\right|^2} \frac{x_0}{\sqrt{2}} \left(\sigma + \sigma^* \right) \\
&= \frac{\sqrt{2} x_0 \operatorname{Re} \sigma}{1 + \left|\sigma\right|^2}.
\end{aligned} \tag{1.4}$$

Any imaginary component in σ will reduce the expectation, so we are constrained to picking a real value.

The derivative of

$$f(\sigma) = \frac{\sigma}{1+\sigma^2}, \tag{1.5}$$

is

$$f'(\sigma) = \frac{1-\sigma^2}{(1+\sigma^2)^2}. \tag{1.6}$$

That has zeros at $\sigma = \pm 1$. The second derivative is

$$f''(\sigma) = \frac{-2\sigma(3-\sigma^2)}{(1+\sigma^2)^3}. \tag{1.7}$$

That will be negative (maximum for the extreme value) at $\sigma = 1$, so the linear superposition of these first two energy eigenkets that maximizes the position expectation is

$$\psi = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right). \tag{1.8}$$

That maximized position expectation is

$$\langle x \rangle = \frac{x_0}{\sqrt{2}}. \tag{1.9}$$

Part 2. The time evolution is given by

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-iHt/\hbar}\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-i(0+1/2)\hbar\omega t/\hbar}|0\rangle + e^{-i(1+1/2)\hbar\omega t/\hbar}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right).
\end{aligned} \tag{1.10}$$

Part 3. The position expectation in the Schrödinger representation is

$$\begin{aligned}
\langle x(t) \rangle &= \frac{1}{2}\left(e^{i\omega t/2}\langle 0| + e^{3i\omega t/2}\langle 1|\right)\frac{x_0}{\sqrt{2}}\left(a^\dagger + a\right)\left(e^{-i\omega t/2}|0\rangle + e^{-3i\omega t/2}|1\rangle\right) \\
&= \frac{x_0}{2\sqrt{2}}\left(e^{i\omega t/2}\langle 0| + e^{3i\omega t/2}\langle 1|\right)\left(e^{-i\omega t/2}|1\rangle + e^{-3i\omega t/2}\sqrt{2}|2\rangle + e^{-3i\omega t/2}|0\rangle\right) \\
&= \frac{x_0}{\sqrt{2}}\cos(\omega t).
\end{aligned} \tag{1.11}$$

Part 4.

$$
\begin{aligned}
\langle x(t)\rangle &= \frac{1}{2}\left(\langle 0| + \langle 1|\right)\frac{x_0}{\sqrt{2}}\left(a^\dagger e^{i\omega t} + a e^{-i\omega t}\right)\left(|0\rangle + |1\rangle\right) \\
&= \frac{x_0}{2\sqrt{2}}\left(\langle 0| + \langle 1|\right)\left(e^{i\omega t}|1\rangle + \sqrt{2}e^{i\omega t}|2\rangle + e^{-i\omega t}|0\rangle\right) \\
&= \frac{x_0}{\sqrt{2}}\cos(\omega t),
\end{aligned}
\tag{1.12}
$$

matching the calculation using the Schrödinger picture.

Part 5. Let's use the Heisenberg picture for the uncertainty calculation. Using the calculation above we have

$$
\begin{aligned}
\langle x^2\rangle &= \frac{1}{2}\frac{x_0^2}{2}\left(e^{-i\omega t}\langle 1| + \sqrt{2}e^{-i\omega t}\langle 2| + e^{i\omega t}\langle 0|\right)\left(e^{i\omega t}|1\rangle + \sqrt{2}e^{i\omega t}|2\rangle + e^{-i\omega t}|0\rangle\right) \\
&= \frac{x_0^2}{4}(1+2+1) \\
&= x_0^2.
\end{aligned}
\tag{1.13}
$$

The uncertainty is

$$
\begin{aligned}
\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 \\
&= x_0^2 - \frac{x_0^2}{2}\cos^2(\omega t) \\
&= \frac{x_0^2}{2}\left(2 - \cos^2(\omega t)\right) \\
&= \frac{x_0^2}{2}\left(1 + \sin^2(\omega t)\right)
\end{aligned}
\tag{1.14}
$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1