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Constant magnetic solenoid field

In [2] the following vector potential

$$\mathbf{A} = \frac{B\rho_a^2}{2\rho}\hat{\boldsymbol{\phi}}, \tag{1.1}$$

is introduced in a discussion on the Aharonov-Bohm effect, for configurations where the interior field of a solenoid is either a constant $\mathbf{B}$ or zero.

I wasn't able to make sense of this since the field I was calculating was zero for all $\rho \ne 0$

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&= \left(\hat{\boldsymbol{\rho}}\partial_\rho + \hat{\mathbf{z}}\partial_z + \frac{\hat{\boldsymbol{\phi}}}{\rho}\partial_\phi \right) \times \frac{B\rho_a^2}{2\rho}\hat{\boldsymbol{\phi}} \\
&= \left(\hat{\boldsymbol{\rho}}\partial_\rho + \frac{\hat{\boldsymbol{\phi}}}{\rho}\partial_\phi \right) \times \frac{B\rho_a^2}{2\rho}\hat{\boldsymbol{\phi}} \\
&= \frac{B\rho_a^2}{2}\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}\partial_\rho \left(\frac{1}{\rho}\right) + \frac{B\rho_a^2}{2\rho}\frac{\hat{\boldsymbol{\phi}}}{\rho} \times \partial_\phi \hat{\boldsymbol{\phi}} \\
&= \frac{B\rho_a^2}{2\rho^2}\left(-\hat{\mathbf{z}} + \hat{\boldsymbol{\phi}} \times \partial_\phi \hat{\boldsymbol{\phi}} \right).
\end{aligned} \tag{1.2}$$

Note that the ρ partial requires that $\rho \ne 0$. To expand the cross product in the second term let $j = \mathbf{e}_1\mathbf{e}_2$, and expand using a Geometric Algebra representation of the unit vector

$$\begin{aligned}
\hat{\boldsymbol{\phi}} \times \partial_\phi \hat{\boldsymbol{\phi}} &= \mathbf{e}_2 e^{j\phi} \times \left(\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 e^{j\phi} \right) \\
&= -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \left\langle \mathbf{e}_2 e^{j\phi} (-\mathbf{e}_1) e^{j\phi} \right\rangle_2 \\
&= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 \\
&= \mathbf{e}_3 \\
&= \hat{\mathbf{z}}.
\end{aligned} \tag{1.3}$$

So, provided $\rho \ne 0$, $\mathbf{B} = 0$.

The errata [1] provides the clarification, showing that a $\rho > \rho_a$ constraint is required for this potential to produce the desired results. Continuity at $\rho = \rho_a$ means that in the interior (or at least on the boundary) we must have one of

$$\mathbf{A} = \frac{B\rho_a}{2}\hat{\boldsymbol{\phi}}, \tag{1.4}$$

or

$$\mathbf{A} = \frac{B\rho}{2}\hat{\boldsymbol{\phi}}. \tag{1.5}$$

The first doesn't work, but the second does

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&= \left(\hat{\boldsymbol{\rho}}\partial_\rho + \hat{\mathbf{z}}\partial_z + \frac{\hat{\boldsymbol{\phi}}}{\rho}\partial_\phi\right) \times \frac{B\rho}{2}\hat{\boldsymbol{\phi}} \\
&= \frac{B}{2}\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} + \frac{B\rho}{2}\frac{\hat{\boldsymbol{\phi}}}{\rho} \times \partial_\phi\hat{\boldsymbol{\phi}} \\
&= B\hat{\mathbf{z}}.
\end{aligned} \tag{1.6}$$

So the vector potential that we want for a constant $B\hat{\mathbf{z}}$ field in the interior $\rho < \rho_a$ of a cylindrical space, we need

$$\mathbf{A} = \begin{cases} \frac{B\rho_a^2}{2\rho}\hat{\boldsymbol{\phi}} & \text{if } \rho \ge \rho_a \\ \frac{B\rho}{2}\hat{\boldsymbol{\phi}} & \text{if } \rho \le \rho_a. \end{cases} \tag{1.7}$$

This potential is graphed in fig. 1.1.

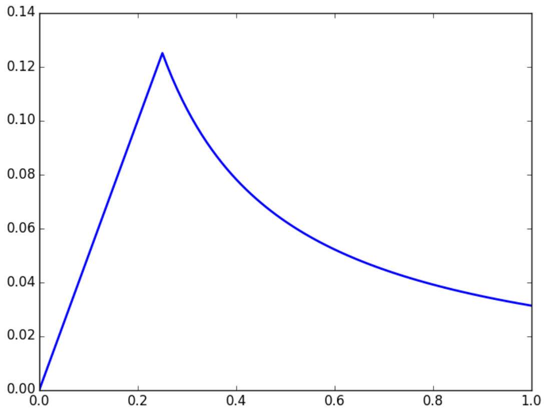

Figure 1.1: Vector potential for constant field in cylindrical region.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Errata: Typographical Errors, Mistakes, and Comments, Modern Quantum Mechanics, 2nd Edition*, 2013. URL http://www.rpi.edu/dept/phys/Courses/PHYS6520/Spring2015/ErrataMQM.pdf. 1

[2] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1