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Some spin problems

Problems from angular momentum chapter of [1].

Q: S_y eigenvectors Find the eigenvectors of σ_y, and then find the probability that a measurement of S_y will be $\hbar/2$ when the state is initially

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{1.1}$$

A: The eigenvalues should be ± 1, which is easily checked

$$\begin{aligned} 0 &= \left| \sigma_y - \lambda \right| \\ &= \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} \\ &= \lambda^2 - 1. \end{aligned} \tag{1.2}$$

For $|+\rangle = (a, b)^\mathrm{T}$ we must have

$$-1a - ib = 0, \tag{1.3}$$

so

$$|+\rangle \propto \begin{bmatrix} -i \\ 1 \end{bmatrix}, \tag{1.4}$$

or

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}. \tag{1.5}$$

For $|-\rangle$ we must have

$$a - ib = 0, \tag{1.6}$$

so

$$|+\rangle \propto \begin{bmatrix} i \\ 1 \end{bmatrix}, \tag{1.7}$$

or

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{1.8}$$

The normalized eigenvectors are

$$\boxed{|\pm\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}.} \tag{1.9}$$

For the probability question we are interested in

$$\begin{aligned} \left| \langle S_y; + | \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right|^2 &= \frac{1}{2} \left| \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right|^2 \\ &= \frac{1}{2} \left(|\alpha|^2 + |\beta|^2 \right) \\ &= \frac{1}{2}. \end{aligned} \tag{1.10}$$

There is a 50% chance of finding the particle in the $|S_x; +\rangle$ state, independent of the initial state.

Q: Magnetic Hamiltonian eigenvectors Using Pauli matrices, find the eigenvectors for the magnetic spin interaction Hamiltonian

$$H = -\frac{1}{\hbar} 2\mu \mathbf{S} \cdot \mathbf{B}. \tag{1.11}$$

A:

$$\begin{aligned} H &= -\mu \boldsymbol{\sigma} \cdot \mathbf{B} \\ &= -\mu \left(B_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + B_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + B_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \\ &= -\mu \begin{bmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{bmatrix}. \end{aligned} \tag{1.12}$$

The characteristic equation is

$$\begin{aligned} 0 &= \begin{vmatrix} -\mu B_z - \lambda & -\mu(B_x - iB_y) \\ -\mu(B_x + iB_y) & \mu B_z - \lambda \end{vmatrix} \\ &= -\left((\mu B_z)^2 - \lambda^2 \right) - \mu^2 \left(B_x^2 - (iB_y)^2 \right) \\ &= \lambda^2 - \mu^2 \mathbf{B}^2. \end{aligned} \tag{1.13}$$

That is

$$\boxed{\lambda = \pm \mu B.} \tag{1.14}$$

Now for the eigenvectors. We are looking for $|\pm\rangle = (a, b)^{\mathrm{T}}$ such that

$$0 = (-\mu B_z \mp \mu B) a - \mu (B_x - iB_y) b \tag{1.15}$$

or

$$|\pm\rangle \propto \begin{bmatrix} B_x - iB_y \\ B_z \pm B \end{bmatrix}. \tag{1.16}$$

This squares to

$$B_x^2 + B_y^2 + B_z^2 + B^2 \pm 2BB_z = 2B(B \pm B_z), \tag{1.17}$$

so the normalized eigenkets are

$$\boxed{|\pm\rangle = \frac{1}{\sqrt{2B(B \pm B_z)}} \begin{bmatrix} B_x - iB_y \\ B_z \pm B \end{bmatrix}.} \tag{1.18}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1