Peeter Joot
peeterjoot@protonmail.com

Spin three halves spin interaction

Exercise 1.1 **Spin three halves spin interaction.** ([1] *pr. 3.33*)

A spin 3/2 nucleus subjected to an external electric field has an interaction Hamiltonian of the form

$$H = \frac{eQ}{2s(s-1)\hbar^2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 S_x^2 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 S_y^2 + \left(\frac{\partial^2 \phi}{\partial z^2} \right)_0 S_z^2 \right). \tag{1.1}$$

1. Show that the interaction energy can be written as

$$A(3S_z^2 - \mathbf{S}^2) + B(S_+^2 + S_-^2). \tag{1.2}$$

2. Find the energy eigenvalues for such a Hamiltonian.

Answer for Exercise 1.1

Part 1. Reordering

$$\begin{aligned} S_+ &= S_x + i S_y \\ S_- &= S_x - i S_y, \end{aligned} \tag{1.3}$$

gives

$$\begin{aligned} S_x &= \frac{1}{2} \left(S_+ + S_- \right) \\ S_y &= \frac{1}{2i} \left(S_+ - S_- \right). \end{aligned} \tag{1.4}$$

The squared spin operators are

$$\begin{aligned} S_x^2 &= \frac{1}{4} \left(S_+^2 + S_-^2 + S_+ S_- + S_- S_+ \right) \\ &= \frac{1}{4} \left(S_+^2 + S_-^2 + 2(S_x^2 + S_y^2) \right) \\ &= \frac{1}{4} \left(S_+^2 + S_-^2 + 2(\mathbf{S}^2 - S_z^2) \right), \end{aligned} \tag{1.5}$$

$$\begin{aligned} S_y^2 &= -\frac{1}{4}\left(S_+^2 + S_-^2 - S_+S_- - S_-S_+\right) \\ &= -\frac{1}{4}\left(S_+^2 + S_-^2 - 2(S_x^2 + S_y^2)\right) \\ &= -\frac{1}{4}\left(S_+^2 + S_-^2 - 2(\mathbf{S}^2 - S_z^2)\right). \end{aligned} \tag{1.6}$$

This gives

$$\begin{aligned} H = \frac{eQ}{2s(s-1)\hbar^2}\Bigg(&\frac{1}{4}\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 \left(S_+^2 + S_-^2 + 2(\mathbf{S}^2 - S_z^2)\right) \\ &- \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 \left(S_+^2 + S_-^2 - 2(\mathbf{S}^2 - S_z^2)\right) \\ &+ \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 S_z^2\Bigg) \\ = \frac{eQ}{2s(s-1)\hbar^2}\Bigg(&\frac{1}{4}\left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right)\left(S_+^2 + S_-^2\right) \\ &+ \frac{1}{2}\left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right)\mathbf{S}^2 \\ &+ \left(\left(\frac{\partial^2\phi}{\partial z^2}\right)_0 - \frac{1}{2}\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \frac{1}{2}\left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right) S_z^2\Bigg). \end{aligned} \tag{1.7}$$

For a static electric field we have

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \tag{1.8}$$

but are evaluating it at a point away from the generating charge distribution, so $\nabla^2\phi = 0$ at that point. This gives

$$\begin{aligned} H = \frac{eQ}{4s(s-1)\hbar^2}\Bigg(&\frac{1}{2}\left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right)\left(S_+^2 + S_-^2\right) \\ &+ \left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right)\left(\mathbf{S}^2 - 3S_z^2\right)\Bigg), \end{aligned} \tag{1.9}$$

so

$$A = -\frac{eQ}{4s(s-1)\hbar^2}\left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right) \tag{1.10}$$

$$B = \frac{eQ}{8s(s-1)\hbar^2}\left(\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 - \left(\frac{\partial^2\phi}{\partial y^2}\right)_0\right). \tag{1.11}$$

Part 2. Using sakuraiProblem3.33.nb, matrix representations for the spin three halves operators and the Hamiltonian were constructed with respect to the basis $\{|3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle\}$

$$
\begin{aligned}
S_+ &= \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
S_- &= \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \\
S_x &= \hbar \begin{bmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{bmatrix} \\
S_y &= i\hbar \begin{bmatrix} 0 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{bmatrix} \\
S_z &= \frac{\hbar}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\
H &= \begin{bmatrix} 3A & 0 & 2\sqrt{3}B & 0 \\ 0 & -3A & 0 & 2\sqrt{3}B \\ 2\sqrt{3}B & 0 & -3A & 0 \\ 0 & 2\sqrt{3}B & 0 & 3A \end{bmatrix}.
\end{aligned}
\tag{1.12}
$$

The energy eigenvalues were found to be

$$
E = \pm\hbar^2\sqrt{9A^2 + 12B^2}, \tag{1.13}
$$

with two fold degeneracies for each eigenvalue.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1