

Spin three halves spin interaction

Exercise 1.1 Spin three halves spin interaction. ([1] pr. 3.33)

A spin 3/2 nucleus subjected to an external electric field has an interaction Hamiltonian of the form

$$H = \frac{eQ}{2s(s-1)\hbar^2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 S_x^2 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 S_y^2 + \left(\frac{\partial^2 \phi}{\partial z^2} \right)_0 S_z^2 \right). \quad (1.1)$$

1. Show that the interaction energy can be written as

$$A(3S_z^2 - \mathbf{S}^2) + B(S_+^2 + S_-^2). \quad (1.2)$$

2. Find the energy eigenvalues for such a Hamiltonian.

Answer for Exercise 1.1

Part 1. Reordering

$$\begin{aligned} S_+ &= S_x + iS_y \\ S_- &= S_x - iS_y, \end{aligned} \quad (1.3)$$

gives

$$\begin{aligned} S_x &= \frac{1}{2}(S_+ + S_-) \\ S_y &= \frac{1}{2i}(S_+ - S_-). \end{aligned} \quad (1.4)$$

The squared spin operators are

$$\begin{aligned} S_x^2 &= \frac{1}{4}(S_+^2 + S_-^2 + S_+S_- + S_-S_+) \\ &= \frac{1}{4}(S_+^2 + S_-^2 + 2(S_x^2 + S_y^2)) \\ &= \frac{1}{4}(S_+^2 + S_-^2 + 2(\mathbf{S}^2 - S_z^2)), \end{aligned} \quad (1.5)$$

$$\begin{aligned}
S_y^2 &= -\frac{1}{4} (S_+^2 + S_-^2 - S_+ S_- - S_- S_+) \\
&= -\frac{1}{4} (S_+^2 + S_-^2 - 2(S_x^2 + S_y^2)) \\
&= -\frac{1}{4} (S_+^2 + S_-^2 - 2(\mathbf{S}^2 - S_z^2)).
\end{aligned} \tag{1.6}$$

This gives

$$\begin{aligned}
H &= \frac{eQ}{2s(s-1)\hbar^2} \left(\frac{1}{4} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 (S_+^2 + S_-^2 + 2(\mathbf{S}^2 - S_z^2)) \right. \\
&\quad \left. - \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 (S_+^2 + S_-^2 - 2(\mathbf{S}^2 - S_z^2)) \right. \\
&\quad \left. + \left(\frac{\partial^2 \phi}{\partial z^2} \right)_0 S_z^2 \right) \\
&= \frac{eQ}{2s(s-1)\hbar^2} \left(\frac{1}{4} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 - \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) (S_+^2 + S_-^2) \right. \\
&\quad \left. + \frac{1}{2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) \mathbf{S}^2 \right. \\
&\quad \left. + \left(\left(\frac{\partial^2 \phi}{\partial z^2} \right)_0 - \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 - \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) S_z^2 \right).
\end{aligned} \tag{1.7}$$

For a static electric field we have

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \tag{1.8}$$

but are evaluating it at a point away from the generating charge distribution, so $\nabla^2 \phi = 0$ at that point. This gives

$$\begin{aligned}
H &= \frac{eQ}{4s(s-1)\hbar^2} \left(\frac{1}{2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 - \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) (S_+^2 + S_-^2) \right. \\
&\quad \left. + \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) (\mathbf{S}^2 - 3S_z^2) \right),
\end{aligned} \tag{1.9}$$

so

$$A = -\frac{eQ}{4s(s-1)\hbar^2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right) \tag{1.10}$$

$$B = \frac{eQ}{8s(s-1)\hbar^2} \left(\left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 - \left(\frac{\partial^2 \phi}{\partial y^2} \right)_0 \right). \tag{1.11}$$

Part 2. Using sakuraiProblem3.33.nb, matrix representations for the spin three halves operators and the Hamiltonian were constructed with respect to the basis $\{|3/2\rangle, |1/2\rangle, |-1/2\rangle, |-3/2\rangle\}$

$$\begin{aligned}
 S_+ &= \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 S_- &= \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \\
 S_x &= \hbar \begin{bmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{bmatrix} \\
 S_y &= i\hbar \begin{bmatrix} 0 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{bmatrix} \\
 S_z &= \frac{\hbar}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\
 H &= \begin{bmatrix} 3A & 0 & 2\sqrt{3}B & 0 \\ 0 & -3A & 0 & 2\sqrt{3}B \\ 2\sqrt{3}B & 0 & -3A & 0 \\ 0 & 2\sqrt{3}B & 0 & 3A \end{bmatrix}.
 \end{aligned} \tag{1.12}$$

The energy eigenvalues were found to be

$$E = \pm \hbar^2 \sqrt{9A^2 + 12B^2}, \tag{1.13}$$

with two fold degeneracies for each eigenvalue.

Bibliography

- [1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. [1.1](#)