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Time evolution of spin half probability and dispersion

Exercise 1.1 Time evolution of spin half probability and dispersion ([1] *pr. 2.3*)

A spin 1/2 system $\mathbf{S} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}} = \sin\beta \hat{\mathbf{x}} + \cos\beta \hat{\mathbf{z}}$, is in state with eigenvalue $\hbar/2$, acted on by a magnetic field of strength B in the $+z$ direction.

1. If S_x is measured at time t, what is the probability of getting $+\hbar/2$?
2. Evaluate the dispersion in S_x as a function of t, that is,

$$\left\langle \left(S_x - \left\langle S_x \right\rangle\right)^2 \right\rangle. \tag{1.1}$$

3. Check your answers for $\beta \rightarrow 0, \pi/2$ to see if they make sense.

Answer for Exercise 1.1

Part 1. The spin operator in matrix form is

$$\begin{aligned} S \cdot \hat{\mathbf{n}} &= \frac{\hbar}{2} \left(\sigma_z \cos\beta + \sigma_x \sin\beta \right) \\ &= \frac{\hbar}{2} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cos\beta + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sin\beta \right) \\ &= \frac{\hbar}{2} \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}. \end{aligned} \tag{1.2}$$

The $\left| S \cdot \hat{\mathbf{n}}; + \right\rangle$ eigenstate is found from

$$\left(S \cdot \hat{\mathbf{n}} - \hbar/2 \right) \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{1.3}$$

or

$$\begin{aligned} 0 &= \left(\cos\beta - 1 \right) a + \sin\beta b \\ &= \left(-2\sin^2(\beta/2) \right) a + 2\sin(\beta/2)\cos(\beta/2) b \\ &= \left(-\sin(\beta/2) \right) a + \cos(\beta/2) b, \end{aligned} \tag{1.4}$$

or

$$|S \cdot \hat{\mathbf{n}}; +\rangle = \begin{bmatrix} \cos(\beta/2) \\ \sin(\beta/2) \end{bmatrix}. \tag{1.5}$$

The Hamiltonian is

$$H = -\frac{eB}{mc} S_z = -\frac{eB\hbar}{2mc} \sigma_z, \tag{1.6}$$

so the time evolution operator is

$$\begin{aligned} U &= e^{-iHt/\hbar} \\ &= e^{\frac{ieBt}{2mc}\sigma_z}. \end{aligned} \tag{1.7}$$

Let $\omega = eB/(2mc)$, so

$$\begin{aligned} U &= e^{i\sigma_z \omega t} \\ &= \cos(\omega t) + i\sigma_z \sin(\omega t) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos(\omega t) + i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sin(\omega t) \\ &= \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix}. \end{aligned} \tag{1.8}$$

The time evolution of the initial state is

$$\begin{aligned} |S \cdot \hat{\mathbf{n}}; +\rangle (t) &= U |S \cdot \hat{\mathbf{n}}; +\rangle (0) \\ &= \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} \cos(\beta/2) \\ \sin(\beta/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\beta/2) e^{i\omega t} \\ \sin(\beta/2) e^{-i\omega t} \end{bmatrix}. \end{aligned} \tag{1.9}$$

The probability of finding the state in $|S \cdot \hat{\mathbf{x}}; +\rangle$ at time t (i.e. measuring S_x and finding $\hbar/2$) is

$$\begin{aligned} |\langle S \cdot \hat{\mathbf{x}}; + | S \cdot \hat{\mathbf{n}}; +\rangle|^2 &= \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(\beta/2) e^{i\omega t} \\ \sin(\beta/2) e^{-i\omega t} \end{bmatrix} \right|^2 \\ &= \frac{1}{2} \left| \cos(\beta/2) e^{i\omega t} + \sin(\beta/2) e^{-i\omega t} \right|^2 \\ &= \frac{1}{2} \left(1 + 2\cos(\beta/2)\sin(\beta/2)\cos(2\omega t)\right) \\ &= \frac{1}{2} \left(1 + \sin(\beta)\cos(2\omega t)\right). \end{aligned} \tag{1.10}$$

Part 2. To calculate the dispersion first note that

$$\begin{aligned} S_x^2 &= \left(\frac{\hbar}{2}\right)^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \\ &= \left(\frac{\hbar}{2}\right)^2, \end{aligned} \tag{1.11}$$

so only the first order expectation is non-trivial to calculate. That is

$$\begin{aligned} \langle S_x \rangle &= \frac{\hbar}{2} \begin{bmatrix} \cos(\beta/2)e^{-i\omega t} & \sin(\beta/2)e^{i\omega t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\beta/2)e^{i\omega t} \\ \sin(\beta/2)e^{-i\omega t} \end{bmatrix} \\ &= \frac{\hbar}{2} \begin{bmatrix} \cos(\beta/2)e^{-i\omega t} & \sin(\beta/2)e^{i\omega t} \end{bmatrix} \begin{bmatrix} \sin(\beta/2)e^{-i\omega t} \\ \cos(\beta/2)e^{i\omega t} \end{bmatrix} \\ &= \frac{\hbar}{2} \sin(\beta/2)\cos(\beta/2) \left(e^{-2i\omega t} + e^{2i\omega t} \right) \\ &= \frac{\hbar}{2} \sin\beta \cos(2\omega t). \end{aligned} \tag{1.12}$$

This gives

$$\boxed{\left\langle (\Delta S_x)^2 \right\rangle = \left(\frac{\hbar}{2}\right)^2 \left(1 - \sin^2\beta \cos^2(2\omega t)\right)} \tag{1.13}$$

Part 3. For $\beta = 0$, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, and $\beta = \pi/2$, $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. For the first case, the state is in an eigenstate of S_z, so must evolve as

$$|S \cdot \hat{\mathbf{n}}; +\rangle (t) = |S \cdot \hat{\mathbf{n}}; +\rangle (0) e^{i\omega t}. \tag{1.14}$$

The probability of finding it in state $|S \cdot \hat{\mathbf{x}}; +\rangle$ is therefore

$$\begin{aligned} &\left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\omega t} \\ 0 \end{bmatrix} \right|^2 \\ &= \frac{1}{2} \left| e^{i\omega t} \right|^2 \\ &= \frac{1}{2} \\ &= \frac{1}{2} \left(1 + \sin(0)\cos(2\omega t)\right). \end{aligned} \tag{1.15}$$

This matches eq. (1.10) as expected.
For $\beta = \pi/2$ we have

$$\begin{aligned} |S \cdot \hat{\mathbf{x}}; +\rangle (t) &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{bmatrix}. \end{aligned} \tag{1.16}$$

The probability for the $\hbar/2$ S_x measurement at time t is

$$
\begin{aligned}
\left| \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{bmatrix} \right|^2 &= \frac{1}{4} \left| e^{i\omega t} + e^{-i\omega t} \right|^2 \\
&= \cos^2(\omega t) \\
&= \frac{1}{2} \left(1 + \sin(\pi/2) \cos(2\omega t) \right) .
\end{aligned}
\tag{1.17}
$$

Again, this matches the expected value.
For the dispersions, at $\beta = 0$, the dispersion is

$$
\left(\frac{\hbar}{2} \right)^2
\tag{1.18}
$$

This is the maximum dispersion, which makes sense since we are measuring S_x when the initial state is $|S \cdot \hat{\mathbf{z}}; +\rangle$. For $\beta = \pi/2$ the dispersion is

$$
\left(\frac{\hbar}{2} \right)^2 \sin^2(2\omega t).
\tag{1.19}
$$

This starts off as zero dispersion (because the initial state is $|S \cdot \hat{\mathbf{x}}; +\rangle$, but then oscillates.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1