Peeter Joot
peeterjoot@protonmail.com

Commutators for some symmetry operators

Q: [1] pr 4.2 If $\mathcal{T}_{\mathbf{d}}$, $\mathcal{D}(\hat{\mathbf{n}}, \phi)$, and π denote the translation, rotation, and parity operators respectively. Which of the following commute and why

(a) $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}'}$, translations in different directions.

(b) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{D}(\hat{\mathbf{n}}', \phi')$, rotations in different directions.

(c) $\mathcal{T}_{\mathbf{d}}$ and π.

(d) $\mathcal{D}(\hat{\mathbf{n}}, \phi$ and π.

A: (a) Consider

$$\begin{aligned}\mathcal{T}_{\mathbf{d}}\mathcal{T}_{\mathbf{d}'} \left|\mathbf{x}\right\rangle &= \mathcal{T}_{\mathbf{d}} \left|\mathbf{x} + \mathbf{d}'\right\rangle \\ &= \left|\mathbf{x} + \mathbf{d}' + \mathbf{d}\right\rangle ,\end{aligned} \tag{1.1}$$

and the reverse application of the translation operators

$$\begin{aligned}\mathcal{T}_{\mathbf{d}'}\mathcal{T}_{\mathbf{d}} \left|\mathbf{x}\right\rangle &= \mathcal{T}_{\mathbf{d}'} \left|\mathbf{x} + \mathbf{d}\right\rangle \\ &= \left|\mathbf{x} + \mathbf{d} + \mathbf{d}'\right\rangle \\ &= \left|\mathbf{x} + \mathbf{d}' + \mathbf{d}\right\rangle .\end{aligned} \tag{1.2}$$

so we see that

$$[\mathcal{T}_{\mathbf{d}}, \mathcal{T}_{\mathbf{d}'}] \left|\mathbf{x}\right\rangle = 0, \tag{1.3}$$

for any position state $\left|\mathbf{x}\right\rangle$, and therefore in general they commute.

A: (b) That rotations do not commute when they are in different directions (like any two orthogonal directions) need not be belaboured.

A: (c) We have

$$\begin{aligned} \mathcal{T}_{\mathbf{d}} \pi |\mathbf{x}\rangle &= \mathcal{T}_{\mathbf{d}} |-\mathbf{x}\rangle \\ &= |-\mathbf{x} + \mathbf{d}\rangle , \end{aligned} \tag{1.4}$$

yet

$$\begin{aligned} \pi \mathcal{T}_{\mathbf{d}} |\mathbf{x}\rangle &= \pi |\mathbf{x} + \mathbf{d}\rangle \\ &= |-\mathbf{x} - \mathbf{d}\rangle \\ &\neq |-\mathbf{x} + \mathbf{d}\rangle . \end{aligned} \tag{1.5}$$

so, in general $[\mathcal{T}_{\mathbf{d}}, \pi] \neq 0$.

A: (d) We have

$$\begin{aligned} \pi \mathcal{D}(\hat{\mathbf{n}}, \phi) |\mathbf{x}\rangle &= \pi \mathcal{D}(\hat{\mathbf{n}}, \phi) \pi^\dagger \pi |\mathbf{x}\rangle \\ &= \pi \mathcal{D}(\hat{\mathbf{n}}, \phi) \pi^\dagger \pi |\mathbf{x}\rangle \\ &= \pi \left(\sum_{k=0}^{\infty} \frac{(-i\mathbf{J} \cdot \hat{\mathbf{n}})^k}{k!} \right) \pi^\dagger \pi |\mathbf{x}\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-i(\pi \mathbf{J} \pi^\dagger) \cdot (\pi \hat{\mathbf{n}} \pi^\dagger))^k}{k!} \pi |\mathbf{x}\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-i\mathbf{J} \cdot \hat{\mathbf{n}})^k}{k!} \pi |\mathbf{x}\rangle \\ &= \mathcal{D}(\hat{\mathbf{n}}, \phi) \pi |\mathbf{x}\rangle , \end{aligned} \tag{1.6}$$

so $[\mathcal{D}(\hat{\mathbf{n}}, \phi), \pi] |\mathbf{x}\rangle = 0$, for any position state $|\mathbf{x}\rangle$, and therefore these operators commute in general.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1