Peeter Joot
peeterjoot@protonmail.com

Plane wave and spinor under time reversal

Q: [1] pr 4.7

(a) Find the time reversed form of a spinless plane wave state in three dimensions.

(b) For the eigenspinor of $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$ expressed in terms of polar and azimuthal angles β and γ, show that $-i\sigma_y \chi^*(\hat{\mathbf{n}})$ has the reversed spin direction.

A: part (a) The Hamiltonian for a plane wave is

$$H = \frac{\mathbf{p}^2}{2m} = i \frac{\partial.}{\partial t} \tag{1.1}$$

Under time reversal the momentum side transforms as

$$\begin{aligned}
\Theta \frac{\mathbf{p}^2}{2m} \Theta^{-1} &= \frac{\left(\Theta \mathbf{p} \Theta^{-1}\right) \cdot \left(\Theta \mathbf{p} \Theta^{-1}\right)}{2m} \\
&= \frac{(-\mathbf{p}) \cdot (-\mathbf{p})}{2m} \\
&= \frac{\mathbf{p}^2}{2m}.
\end{aligned} \tag{1.2}$$

The time derivative side of the equation is also time reversal invariant

$$\begin{aligned}
\Theta i \frac{\partial}{\partial t} \Theta^{-1} &= \Theta i \Theta^{-1} \Theta \frac{\partial}{\partial t} \Theta^{-1} \\
&= -i \frac{\partial}{\partial (-t)} \\
&= i \frac{\partial}{\partial t}.
\end{aligned} \tag{1.3}$$

Solutions to this equation are linear combinations of

$$\psi(\mathbf{x}, t) = e^{i \mathbf{k} \cdot \mathbf{x} - i E t/\hbar}, \tag{1.4}$$

where $\hbar^2 \mathbf{k}^2 / 2m = E$, the energy of the particle. Under time reversal we have

$$
\begin{aligned}
\psi(\mathbf{x},t) &\rightarrow e^{-i\mathbf{k}\cdot\mathbf{x}+iE(-t)/\hbar} \\
&= \left(e^{i\mathbf{k}\cdot\mathbf{x}-iE(-t)/\hbar}\right)^* \\
&= \psi^*(\mathbf{x},-t)
\end{aligned}
\tag{1.5}
$$

A: part (b) The text uses a requirement for time reversal of spin states to show that the Pauli matrix form of the time reversal operator is

$$
\Theta = -i\sigma_y K,
\tag{1.6}
$$

where K is a complex conjugating operator. The form of the spin up state used in that demonstration was

$$
\begin{aligned}
|\hat{\mathbf{n}};+\rangle &= e^{-iS_z\beta/\hbar}e^{-iS_y\gamma/\hbar}\,|+\rangle \\
&= e^{-i\sigma_z\beta/2}e^{-i\sigma_y\gamma/2}\,|+\rangle \\
&= \left(\cos(\beta/2) - i\sigma_z\sin(\beta/2)\right)\left(\cos(\gamma/2) - i\sigma_y\sin(\gamma/2)\right)|+\rangle \\
&= \left(\cos(\beta/2) - i\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\sin(\beta/2)\right)\left(\cos(\gamma/2) - i\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}\sin(\gamma/2)\right)|+\rangle \\
&= \begin{bmatrix}e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2}\end{bmatrix}\begin{bmatrix}\cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2)\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} \\
&= \begin{bmatrix}e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2}\end{bmatrix}\begin{bmatrix}\cos(\gamma/2) \\ \sin(\gamma/2)\end{bmatrix} \\
&= \begin{bmatrix}\cos(\gamma/2)e^{-i\beta/2} \\ \sin(\gamma/2)e^{i\beta/2}\end{bmatrix}.
\end{aligned}
\tag{1.7}
$$

The state orthogonal to this one is claimed to be

$$
\begin{aligned}
|\hat{\mathbf{n}};-\rangle &= e^{-iS_z\beta/\hbar}e^{-iS_y(\gamma+\pi)/\hbar}\,|+\rangle \\
&= e^{-i\sigma_z\beta/2}e^{-i\sigma_y(\gamma+\pi)/2}\,|+\rangle\,.
\end{aligned}
\tag{1.8}
$$

We have

$$
\cos((\gamma+\pi)/2) = \mathrm{Re}\, e^{i(\gamma+\pi)/2} = \mathrm{Re}\, ie^{i\gamma/2} = -\sin(\gamma/2),
\tag{1.9}
$$

and

$$
\sin((\gamma+\pi)/2) = \mathrm{Im}\, e^{i(\gamma+\pi)/2} = \mathrm{Im}\, ie^{i\gamma/2} = \cos(\gamma/2),
\tag{1.10}
$$

so we should have

$$
|\hat{\mathbf{n}};-\rangle = \begin{bmatrix}-\sin(\gamma/2)e^{-i\beta/2} \\ \cos(\gamma/2)e^{i\beta/2}\end{bmatrix}.
\tag{1.11}
$$

This looks right, but we can sanity check orthogonality

$$\langle \hat{\mathbf{n}}; - | \hat{\mathbf{n}}; + \rangle = \begin{bmatrix} -\sin(\gamma/2)e^{i\beta/2} & \cos(\gamma/2)e^{-i\beta/2} \end{bmatrix} \begin{bmatrix} \cos(\gamma/2)e^{-i\beta/2} \\ \sin(\gamma/2)e^{i\beta/2} \end{bmatrix} = 0, \tag{1.12}$$

as expected.

The task at hand appears to be the operation on the column representation of $|\hat{\mathbf{n}}; +\rangle$ using the Pauli representation of the time reversal operator. That is

$$\begin{aligned} \Theta |\hat{\mathbf{n}}; +\rangle &= -i\sigma_y K \begin{bmatrix} e^{-i\beta/2}\cos(\gamma/2) \\ e^{i\beta/2}\sin(\gamma/2) \end{bmatrix} \\ &= -i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{i\beta/2}\cos(\gamma/2) \\ e^{-i\beta/2}\sin(\gamma/2) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\beta/2}\cos(\gamma/2) \\ e^{-i\beta/2}\sin(\gamma/2) \end{bmatrix} \\ &= \begin{bmatrix} -e^{-i\beta/2}\sin(\gamma/2) \\ e^{i\beta/2}\cos(\gamma/2) \end{bmatrix} \\ &= |\hat{\mathbf{n}}; -\rangle . \qquad \square \end{aligned} \tag{1.13}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1