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Totally asymmetric potential

Q: [1] pr 4.11 (a) Given a time reversal invariant Hamiltonian, show that for any energy eigenket

$$\langle \mathbf{L} \rangle = 0. \tag{1.1}$$

(b) If the wave function of such a state is expanded as

$$\sum_{l,m} F_{lm} Y_{lm}(\theta, \phi), \tag{1.2}$$

what are the phase restrictions on F_{lm}?

A: part (a) For a time reversal invariant Hamiltonian H we have

$$H\Theta = \Theta H. \tag{1.3}$$

If $\left| \psi \right\rangle$ is an energy eigenstate with eigenvalue E, we have

$$\begin{aligned} H\Theta \left| \psi \right\rangle &= \Theta H \left| \psi \right\rangle \\ &= \lambda \Theta \left| \psi \right\rangle, \end{aligned} \tag{1.4}$$

so $\Theta \left| \psi \right\rangle$ is also an eigenvalue of H, so can only differ from $\left| \psi \right\rangle$ by a phase factor. That is

$$\begin{aligned} \left| \psi' \right\rangle &= \Theta \left| \psi \right\rangle \\ &= e^{i\delta} \left| \psi \right\rangle. \end{aligned} \tag{1.5}$$

Now consider the expectation of $\mathbf{L}$ with respect to a time reversed state

$$\begin{aligned} \left\langle \psi' \right| \mathbf{L} \left| \psi' \right\rangle &= \left\langle \psi \right| \Theta^{-1} \mathbf{L} \Theta \left| \psi \right\rangle \\ &= \left\langle \psi \right| (-\mathbf{L}) \left| \psi \right\rangle, \end{aligned} \tag{1.6}$$

however, we also have

$$\begin{aligned} \left\langle \psi' \right| \mathbf{L} \left| \psi' \right\rangle &= \left(\left\langle \psi \right| e^{-i\delta} \right) \mathbf{L} \left(e^{i\delta} \left| \psi \right\rangle \right) \\ &= \left\langle \psi \right| \mathbf{L} \left| \psi \right\rangle, \end{aligned} \tag{1.7}$$

so we have $\left\langle \psi \right| \mathbf{L} \left| \psi \right\rangle = - \left\langle \psi \right| \mathbf{L} \left| \psi \right\rangle$ which is only possible if $\langle \mathbf{L} \rangle = \left\langle \psi \right| \mathbf{L} \left| \psi \right\rangle = 0$.

A: part (b) Consider the expansion of the wave function of a time reversed energy eigenstate

$$\begin{aligned}\langle \mathbf{x}| \Theta |\psi\rangle &= \langle \mathbf{x}| e^{i\delta} |\psi\rangle \\ &= e^{i\delta} \langle \mathbf{x}|\psi\rangle \,,\end{aligned} \tag{1.8}$$

and then consider the same state expanded in the position basis

$$\begin{aligned}\langle \mathbf{x}| \Theta |\psi\rangle &= \langle \mathbf{x}| \Theta \int d^3\mathbf{x}' \left(|\mathbf{x}'\rangle \langle \mathbf{x}'| \right) |\psi\rangle \\ &= \langle \mathbf{x}| \Theta \int d^3\mathbf{x}' \left(\langle \mathbf{x}'|\psi\rangle \right) |\mathbf{x}'\rangle \\ &= \langle \mathbf{x}| \int d^3\mathbf{x}' \left(\langle \mathbf{x}'|\psi\rangle \right)^* \Theta |\mathbf{x}'\rangle \\ &= \langle \mathbf{x}| \int d^3\mathbf{x}' \left(\langle \mathbf{x}'|\psi\rangle \right)^* |\mathbf{x}'\rangle \\ &= \int d^3\mathbf{x}' \left(\langle \mathbf{x}'|\psi\rangle \right)^* \langle \mathbf{x}|\mathbf{x}'\rangle \\ &= \int d^3\mathbf{x}' \langle \psi|\mathbf{x}'\rangle \, \delta(\mathbf{x} - \mathbf{x}') \\ &= \langle \psi|\mathbf{x}\rangle \,.\end{aligned} \tag{1.9}$$

This demonstrates a relationship between the wave function and its complex conjugate

$$\langle \mathbf{x}|\psi\rangle = e^{-i\delta} \langle \psi|\mathbf{x}\rangle \,. \tag{1.10}$$

Now expand the wave function in the spherical harmonic basis

$$\begin{aligned}\langle \mathbf{x}|\psi\rangle &= \int d\Omega \, \langle \mathbf{x}|\hat{\mathbf{n}}\rangle \langle \hat{\mathbf{n}}|\psi\rangle \\ &= \sum_{lm} F_{lm}(r) Y_{lm}(\theta, \phi) \\ &= e^{-i\delta} \left(\sum_{lm} F_{lm}(r) Y_{lm}(\theta, \phi) \right)^* \\ &= e^{-i\delta} \sum_{lm} \left(F_{lm}(r) \right)^* Y_{lm}^*(\theta, \phi) \\ &= e^{-i\delta} \sum_{lm} \left(F_{lm}(r) \right)^* (-1)^m Y_{l,-m}(\theta, \phi) \\ &= e^{-i\delta} \sum_{lm} \left(F_{l,-m}(r) \right)^* (-1)^m Y_{l,m}(\theta, \phi),\end{aligned} \tag{1.11}$$

so the F_{lm} functions are constrained by

$$F_{lm}(r) = e^{-i\delta} \left(F_{l,-m}(r) \right)^* (-1)^m. \tag{1.12}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1