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Translation operator problems

Exercise 1.1 One dimensional translation operator. ([1] pr. 1.28)

1. Evaluate the classical Poisson bracket

$$\left[x, F(p) \right]_{\text{classical}} \tag{1.1}$$

2. Evaluate the commutator

$$\left[x, e^{ipa/\hbar} \right] \tag{1.2}$$

3. Using the result in 2, prove that

$$e^{ipa/\hbar} \left| x' \right\rangle, \tag{1.3}$$

is an eigenstate of the coordinate operator x.

Answer for Exercise 1.1

Part 1.

$$\begin{aligned} \left[x, F(p) \right]_{\text{classical}} &= \frac{\partial x}{\partial x} \frac{\partial F(p)}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial F(p)}{\partial x} \\ &= \frac{\partial F(p)}{\partial p}. \end{aligned} \tag{1.4}$$

Part 2. Having worked backwards through these problems, the answer for this one dimensional problem can be obtained from eq. (1.25) and is

$$\left[x, e^{ipa/\hbar} \right] = a e^{ipa/\hbar}. \tag{1.5}$$

Part 3.

$$x e^{ipa/\hbar} \left| x' \right\rangle = \left(\left[x, e^{ipa/\hbar} \right] e^{ipa/\hbar} x + \right) \left| x' \right\rangle = \left(a e^{ipa/\hbar} + e^{ipa/\hbar} x' \right) \left| x' \right\rangle = \left(a + x' \right) \left| x' \right\rangle. \tag{1.6}$$

This demonstrates that $e^{ipa/\hbar} \left| x' \right\rangle$ is an eigenstate of x with eigenvalue $a + x'$.

Exercise 1.2 **Polynomial commutators.** (*[1] pr. 1.29*)

1. For power series F, G, verify

$$[x_k, G(\mathbf{p})] = i\hbar \frac{\partial G}{\partial p_k}, \qquad [p_k, F(\mathbf{x})] = -i\hbar \frac{\partial F}{\partial x_k}. \tag{1.7}$$

2. Evaluate $\left[x^2, p^2\right]$, and compare to the classical Poisson bracket $\left[x^2, p^2\right]_{\text{classical}}$.

Answer for Exercise 1.2

Part 1. Let

$$\begin{aligned} G(\mathbf{p}) &= \sum_{klm} a_{klm} p_1^k p_2^l p_3^m \\ F(\mathbf{x}) &= \sum_{klm} b_{klm} x_1^k x_2^l x_3^m. \end{aligned} \tag{1.8}$$

It is simpler to work with a specific x_k, say $x_k = y$. The validity of the general result will still be clear doing so. Expanding the commutator gives

$$\begin{aligned} \left[y, G(\mathbf{p})\right] &= \sum_{klm} a_{klm} \left[y, p_1^k p_2^l p_3^m\right] \\ &= \sum_{klm} a_{klm} \left(y p_1^k p_2^l p_3^m - p_1^k p_2^l p_3^m y\right) \\ &= \sum_{klm} a_{klm} \left(p_1^k y p_2^l p_3^m - p_1^k y p_2^l p_3^m\right) \\ &= \sum_{klm} a_{klm} p_1^k \left[y, p_2^l\right] p_3^m. \end{aligned} \tag{1.9}$$

From eq. (1.23), we have $\left[y, p_2^l\right] = l i\hbar p_2^{l-1}$, so

$$\begin{aligned} \left[y, G(\mathbf{p})\right] &= \sum_{klm} a_{klm} p_1^k \left[y, p_2^l\right] \left(l i\hbar p_2^{l-1}\right) p_3^m \\ &= i\hbar \frac{\partial G(\mathbf{p})}{\partial y}. \end{aligned} \tag{1.10}$$

It is straightforward to show that $\left[p, x^l\right] = -l i\hbar x^{l-1}$, allowing for a similar computation of the momentum commutator

$$
\begin{aligned}
\left[p_y, F(\mathbf{x})\right] &= \sum_{klm} b_{klm} \left[p_y, x_1^k x_2^l x_3^m\right] \\
&= \sum_{klm} b_{klm} \left(p_y x_1^k x_2^l x_3^m - x_1^k x_2^l x_3^m p_y\right) \\
&= \sum_{klm} b_{klm} \left(x_1^k p_y x_2^l x_3^m - x_1^k p_y x_2^l x_3^m\right) \\
&= \sum_{klm} b_{klm} x_1^k \left[p_y, x_2^l\right] x_3^m \\
&= \sum_{klm} b_{klm} x_1^k \left(-l i \hbar x_2^{l-1}\right) x_3^m \\
&= -i\hbar \frac{\partial F(\mathbf{x})}{\partial p_y}.
\end{aligned} \tag{1.11}
$$

Part 2. It isn't clear to me how the results above can be used directly to compute $\left[x^2, p^2\right]$. However, when the first term of such a commutator is a mononomial, it can be expanded in terms of an x commutator

$$
\begin{aligned}
\left[x^2, G(\mathbf{p})\right] &= x^2 G - G x^2 \\
&= x\,(xG) - Gx^2 \\
&= x\,([x, G] + Gx) - Gx^2 \\
&= x\,[x, G] + (xG)\,x - Gx^2 \\
&= x\,[x, G] + ([x, G] + \cancel{Gx})\,x - \cancel{Gx^2} \\
&= x\,[x, G] + [x, G]\,x.
\end{aligned} \tag{1.12}
$$

Similarly,

$$
\left[x^3, G(\mathbf{p})\right] = x^2\,[x, G] + x\,[x, G]\,x + [x, G]\,x^2. \tag{1.13}
$$

An induction hypothesis can be formed

$$
\left[x^k, G(\mathbf{p})\right] = \sum_{j=0}^{k-1} x^{k-1-j}\,[x, G]\,x^j, \tag{1.14}
$$

and demonstrated

$$
\begin{aligned}
\left[x^{k+1}, G(\mathbf{p})\right] &= x^{k+1}G - Gx^{k+1} \\
&= x\left(x^k G\right) - Gx^{k+1} \\
&= x\left(\left[x^k, G\right] + Gx^k\right) - Gx^{k+1} \\
&= x\left[x^k, G\right] + (xG)\,x^k - Gx^{k+1} \\
&= x\left[x^k, G\right] + ([x, G] + Gx)\,x^k - Gx^{k+1} \\
&= x\left[x^k, G\right] + [x, G]\,x^k \\
&= x\sum_{j=0}^{k-1} x^{k-1-j}\,[x, G]\,x^j + [x, G]\,x^k \\
&= \sum_{j=0}^{k-1} x^{(k+1)-1-j}\,[x, G]\,x^j + [x, G]\,x^k \\
&= \sum_{j=0}^{k} x^{(k+1)-1-j}\,[x, G]\,x^j. \qquad \square
\end{aligned}
\tag{1.15}
$$

That was a bit overkill for this problem, but may be useful later. Application of this to the problem gives

$$
\begin{aligned}
\left[x^2, p^2\right] &= x\left[x, p^2\right] + \left[x, p^2\right]x \\
&= x i\hbar\frac{\partial p^2}{\partial x} + i\hbar\frac{\partial p^2}{\partial x}x \\
&= x2i\hbar p + 2i\hbar p x \\
&= i\hbar\left(2xp + 2px\right).
\end{aligned}
\tag{1.16}
$$

The classical commutator is

$$
\begin{aligned}
\left[x^2, p^2\right]_{\text{classical}} &= \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} \\
&= 2x2p \\
&= 2xp + 2px.
\end{aligned}
\tag{1.17}
$$

This demonstrates the expected relation between the classical and quantum commutators

$$
\left[x^2, p^2\right] = i\hbar\left[x^2, p^2\right]_{\text{classical}}.
\tag{1.18}
$$

Exercise 1.3 Translation operator and position expectation. ([1] *pr. 1.30*)

The translation operator for a finite spatial displacement is given by

$$J(\mathbf{l}) = \exp\left(-i\mathbf{p}\cdot\mathbf{l}/\hbar\right), \tag{1.19}$$

where $\mathbf{p}$ is the momentum operator.

1. Evaluate

$$[x_i, J(\mathbf{l})]. \tag{1.20}$$

2. Demonstrate how the expectation value $\langle \mathbf{x} \rangle$ changes under translation.

Answer for Exercise 1.3

Part 1. For clarity, let's set $x_i = y$. The general result will be clear despite doing so.

$$\left[y, J(\mathbf{l})\right] = \sum_{k=0} \frac{1}{k!}\left(\frac{-i}{\hbar}\right)\left[y, (\mathbf{p}\cdot\mathbf{l})^k\right]. \tag{1.21}$$

The commutator expands as

$$\begin{aligned}
\left[y, (\mathbf{p}\cdot\mathbf{l})^k\right] + (\mathbf{p}\cdot\mathbf{l})^k y = y\,(\mathbf{p}\cdot\mathbf{l})^k \\
&= y\left(p_x l_x + p_y l_y + p_z l_z\right)(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= \left(p_x l_x y + y p_y l_y + p_z l_z y\right)(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= \left(p_x l_x y + l_y\left(p_y y + i\hbar\right) + p_z l_z y\right)(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= (\mathbf{p}\cdot\mathbf{l})\, y\,(\mathbf{p}\cdot\mathbf{l})^{k-1} + i\hbar l_y\,(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= \cdots \\
&= (\mathbf{p}\cdot\mathbf{l})^{k-1}\, y\,(\mathbf{p}\cdot\mathbf{l})^{k-(k-1)} + (k-1) i\hbar l_y\,(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= (\mathbf{p}\cdot\mathbf{l})^{k}\, y + k i\hbar l_y\,(\mathbf{p}\cdot\mathbf{l})^{k-1}.
\end{aligned} \tag{1.22}$$

In the above expansion, the commutation of y with p_x, p_z has been used. This gives, for $k \neq 0$,

$$\left[y, (\mathbf{p}\cdot\mathbf{l})^k\right] = k i\hbar l_y\,(\mathbf{p}\cdot\mathbf{l})^{k-1}. \tag{1.23}$$

Note that this also holds for the $k = 0$ case, since y commutes with the identity operator. Plugging back into the J commutator, we have

$$\begin{aligned}
\left[y, J(\mathbf{l})\right] &= \sum_{k=1} \frac{1}{k!}\left(\frac{-i}{\hbar}\right) k i\hbar l_y\,(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= l_y \sum_{k=1} \frac{1}{(k-1)!}\left(\frac{-i}{\hbar}\right)(\mathbf{p}\cdot\mathbf{l})^{k-1} \\
&= l_y J(\mathbf{l}).
\end{aligned} \tag{1.24}$$

The same pattern clearly applies with the other x_i values, providing the desired relation.

$$[\mathbf{x}, J(\mathbf{l})] = \sum_{m=1}^{3} \mathbf{e}_m l_m J(\mathbf{l}) = \mathbf{l} J(\mathbf{l}). \tag{1.25}$$

Part 2. Suppose that the translated state is defined as $|\alpha_{\mathbf{l}}\rangle = \mathcal{J}(\mathbf{l})\,|\alpha\rangle$. The expectation value with respect to this state is

$$\begin{aligned} \langle \mathbf{x}' \rangle &= \langle \alpha_{\mathbf{l}} |\, \mathbf{x} \,| \alpha_{\mathbf{l}} \rangle \\ &= \langle \alpha |\, \mathcal{J}^{\dagger}(\mathbf{l}) \mathbf{x} \mathcal{J}(\mathbf{l}) \,| \alpha \rangle \\ &= \langle \alpha |\, \mathcal{J}^{\dagger}(\mathbf{l}) \left(\mathbf{x} \mathcal{J}(\mathbf{l}) \right) | \alpha \rangle \\ &= \langle \alpha |\, \mathcal{J}^{\dagger}(\mathbf{l}) \left(\mathcal{J}(\mathbf{l}) \mathbf{x} + \mathbf{l} \mathcal{J}(\mathbf{l}) \right) | \alpha \rangle \\ &= \langle \alpha |\, \mathcal{J}^{\dagger} \mathcal{J} \mathbf{x} + \mathbf{l} \mathcal{J}^{\dagger} \mathcal{J} \,| \alpha \rangle \\ &= \langle \alpha |\, \mathbf{x} \,| \alpha \rangle + \mathbf{l} \langle \alpha | \alpha \rangle \\ &= \langle \mathbf{x} \rangle + \mathbf{l}. \end{aligned} \tag{1.26}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1, 1.2, 1.3